Exploring the Golden Section with Twenty-First Century Tools: GeoGebra

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ABSTRACT: In this paper we illustrate how learners can discover and explore some geometric figures that embed the golden section using GeoGebra. First, we introduce the problem of dividing a given segment into the golden section. Second, we present a method to solve said problem. Next, we explore properties of the golden rectangle, golden triangle, golden spiral, and golden pentagon. We conclude by suggesting some references to find more appearances of the golden section not only in mathematics, but also in nature and art.

KEYWORDS: Golden section, golden rectangle, golden triangle, golden spiral, golden pentagon, GeoGebra.

1. Introduction

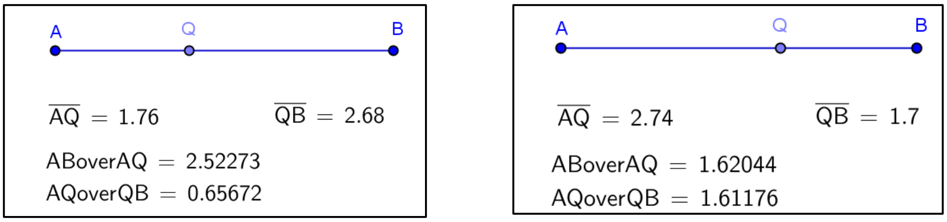
Interactive geometry software such as GeoGebra (GG) allows users and learners to construct effortlessly dynamic diagrams that they can continuously transform. The use of such software facilitates the teaching and learning of properties of mathematical objects, such as numbers and geometric figures.

One of the most ubiquitous numbers is the so called golden number, denoted by the Greek letter φ (phi) in honor to Phidias who used it in the construction of the Parthenon in Athens. The golden number is involved in the solution to the following geometric problem: Given a segment $\overline{AB}$, find an interior point P such that $\frac{AB}{AP} = \frac{AP}{PB}$ (Fig. 1). In other words, point P divides segment $\overline{AB}$, into two segments ($\overline{AB}$ and $\overline{PB}$) such that the ratio of the entire segment to the larger segment is equal to the ratio of the larger segment to the smaller segment. Following Euclid, mathematicians also say that P divides $\overline{AB}$ in extreme and mean ratio. This ratio is also called the golden section, golden mean, golden ratio, golden number, and divide proportion. We will refer to P as the golden point.

Figure 1: Locate a point P on $\overline{AB}$, such that $\frac{AB}{AP} = \frac{AP}{PB}$

2. Estimating the location of the golden point

One of the first questions that we need to ask ourselves is whether P exists. If P exists, we can use GG to estimate dynamically its approximate location. To this end, construct first any point Q on segment $\overline{AB}$ (Fig. 2a). Second, compute the ratios $\frac{AB}{AQ}$ and $\frac{AQ}{QB}$. Next, drag point Q until $\frac{AB}{AQ} \approx \frac{AQ}{QB}$ (Fig. 2b). Notice that $\frac{AB}{AQ}$ is denoted by ABoverAQ (AB over AQ). A similar remark is applicable to $\frac{AQ}{QB}$. and subsequent ratios.



(a) (b)

Figure 2: Estimating the location of Q dynamically

3. Constructing the golden point

Once we have a conjecture that the golden point exists, we need to device a method to construct it. In figure 3 we actually constructed, using only straightedge and compass, point P, the golden point of segment $\overline{AB}$. There are several procedures to construct the golden point. Below we describe one of the simplest methods.

Given segment $\overline{AB}$, construct its midpoint M and then the perpendicular line to $\overline{AB}$ through B (Fig. 3). Next, locate point C on said perpendicular such that BC = MB. Construct segment $\overline{AC}$. Let D be the point of intersection of the circle with center C and radius CB and segment. A circle with center A and radius AD cuts segment $\overline{AB}$ at point P. **P is the golden point.**

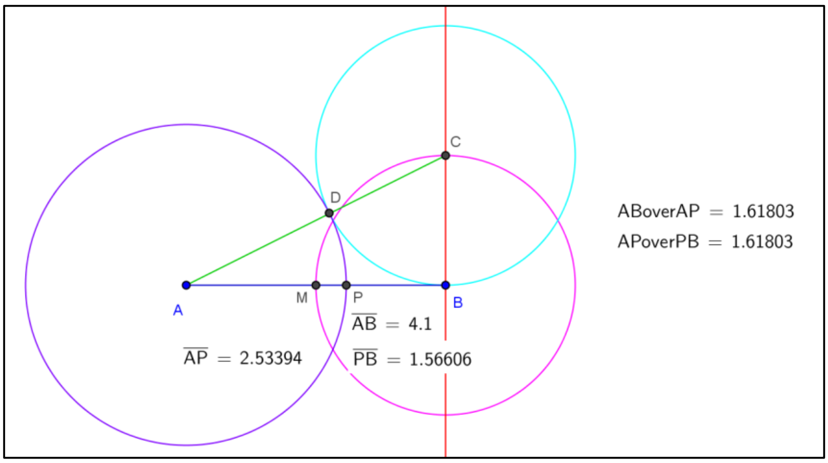


Figure 3: Construction of the golden point using GG

We can now drag any of the endpoints of segment $\overline{AB}$ to verify empirically that point P is indeed the golden point. As figure 4 shows, point P seems to be the golden point.

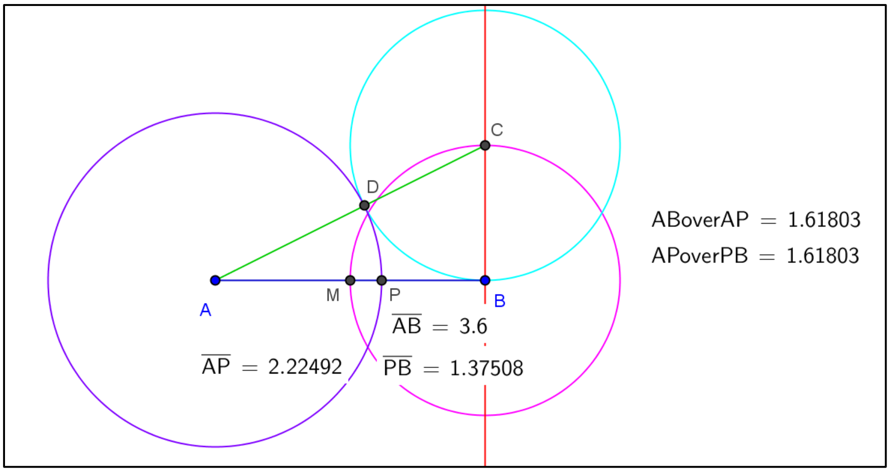


Figure 4: Further verification that P is the golden point

Of course, students should be encouraged to understand why point P divides segment $\overline{AB}$ in the golden section. Below we provide a proof.

Let AB = 2*x*. Then CD = CB = MB = *x*. Applying the Pythagorean Theorem to ΔABC we obtain AC $=\sqrt{5}\,x$, which implies that AP = AD $=\sqrt{5}\,x - x$ and PB = AB – AP = 3*x* – $\sqrt{5}\,x$. We then find that $\frac{AB}{AP}=\frac{2x}{\sqrt{5}x-x} = \frac{2}{\sqrt{5}-1}$ and $\frac{AP}{PB} = \frac{\sqrt{5}x-x}{3x-\sqrt{5}x} = \frac{2}{\sqrt{5}-1}$ because $2(3-\sqrt{5}) = (\sqrt{5}-1)(\sqrt{5}-1)$ or $6-2\sqrt{5} = 6-2\sqrt{5}$. In other words, $\frac{AB}{AP} = \frac{AP}{PB}$.

The ratio $\frac{AB}{AP}\left(\frac{2}{\sqrt{5}-1}\right)$, denoted φ, is known as the golden section. We can easily show that $\varphi = \frac{\sqrt{5}+1}{2} \approx 1.6180339887$. The golden section is very well known for its ubiquitous presence in several geometric figures. Below we describe some appearances of the golden section.

4. Discovering and constructing the golden rectangle

Figure 5 displays a golden rectangle. Students may wonder what makes this appealing shape a golden rectangle. No, it is not because it is colored gold.

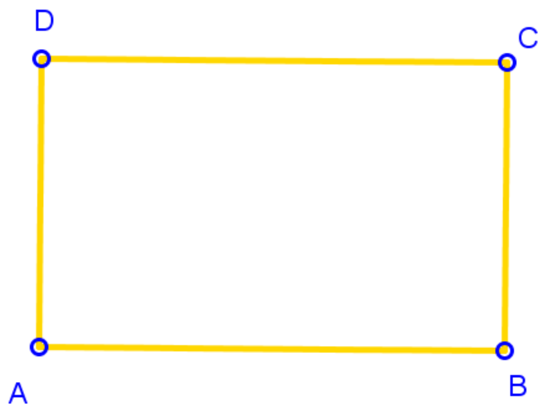


Figure 5: A golden rectangle

Some students may suspect that the golden section is embedded in the golden rectangle. Some of them may carry their guess a little further and conjecture that the golden rectangle is a rectangle whose side lengths are in the ratio φ , $\frac{1+\sqrt{5}}{2}$. They can then use GG and quickly verify that their conjecture seems to be true (Fig. 6).

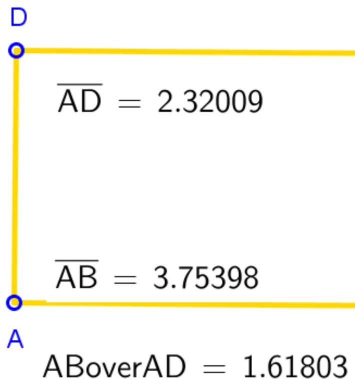


Figure 6: ABCD is a golden rectangle because $\frac{AB}{AD} = \varphi$

As they drag point A, the ratio $\frac{AB}{AD}$, (ABoverAD) seems to be constant. They may wonder, as the authors of this paper did, how complicated the procedure to construct a golden rectangle may be. Surprisingly, nothing could be further from the truth. The following procedure is simple, economical, and elegant.

Construct a square ABCD (Fig. 7). Next, construct the midpoint M of side $\overline{AB}$ and then construct the circle with center M and radius MC. This circle intersects the extension of side AB at point E. Finally, we complete the rectangle AEFD.

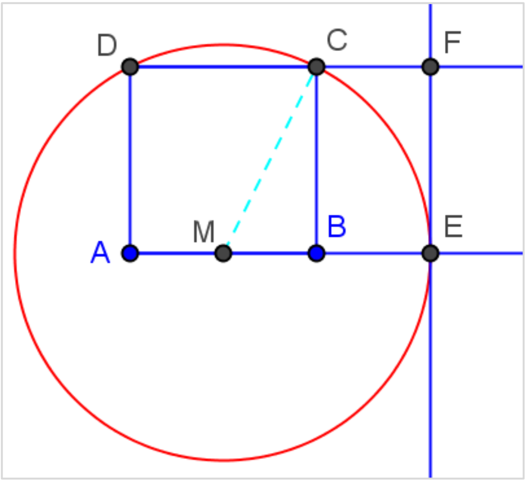


Figure 7: Construction of the golden rectangle using GG

GG has a handy tool that allows us to "construct" regular polygons, in particular squares. As the reader may know, not all regular polygons can be constructed using only straightedge and compass.

The reader may wonder whether this simple construction really produces a golden rectangle. GG effortlessly provides empirical evidence that AEFD seems to be a golden rectangle (Fig. 8).

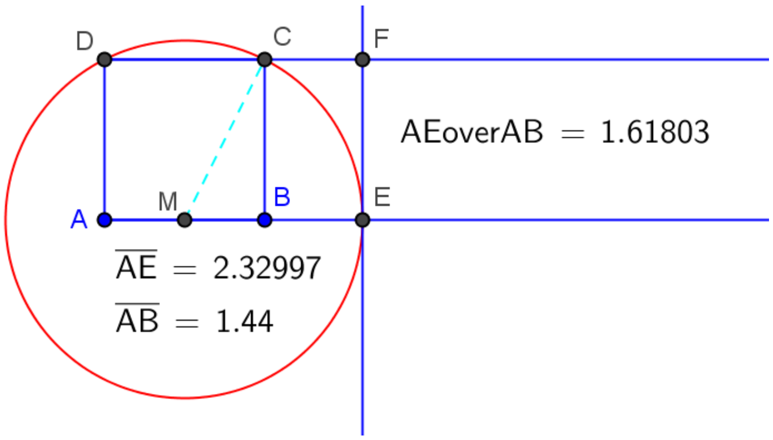


Figure 8: AEFD seems to be a golden rectangle

To justify the claim that AEFD is a golden rectangle we provide the following mathematical argument:

Let AB = 2*x*, so MB = AM = *x* because M is the midpoint of segment $\overline{AB}$ (Fig. 8). Applying the Pythagorean theorem to ΔBCM, we get $MC^2 = MB^2 + BC^2 = x^2 + +4x^2 = 5x^2$; therefore, MC = $\sqrt{5}x$. It follows that AE = AM + ME = AM + MC = $x +\sqrt{5}\ x$. We can now find $\frac{AE}{AD}$, that is, $\frac{AE}{AD} = \frac{AE}{AB} = \frac{x+\sqrt{5}x}{2x} = \frac{1+\sqrt{5}}{2}$=φ. In other words, AEFD is a golden rectangle.

Next, we will show that point B is the golden point of segment $\overline{AE}$. That is, B satisfies $\frac{AE}{AB} = \frac{AB}{BE}$. Since $\frac{AE}{AB}$ = φ, we only need to find $\frac{AB}{BE}$. Using the fact that

BE = AE – AB = $x +\sqrt{5}\ x - 2x = \sqrt{5}\ x - x$ we find that

$\frac{AB}{BE} = \frac{2x}{\sqrt{5}x-x} = \frac{2}{\sqrt{5}-1} = \frac{1+\sqrt{5}}{2}$ =φ.

AEFD is a golden rectangle, but the inquisitive learner may notice that BCFE is also a golden rectangle since $\frac{BC}{BE} = \frac{AB}{BE} = \varphi$. This observation suggests that given a golden rectangle, say AEFD (Fig. 9), we can construct another golden rectangle (BCFE) by constructing a square ABCD whose side is the shortest side of the golden rectangle. We can substantiate our claim mathematically as follows.

Let AB = *x*. Because AEFD is a golden rectangle, we know that $\frac{AE}{AB} = \frac{AE}{AD} = \varphi$, so AE = *x*φ. We want to show that $\frac{BC}{BE}$ = φ. We find that BE = AE – AB = *x*φ – *x*; thus, $\frac{BC}{BE} = \frac{AB}{BE} = \frac{x}{x\varphi-x} = \frac{1}{\varphi-1} = \frac{1+\sqrt{5}}{2}$. In other words, BCFE is a golden rectangle.

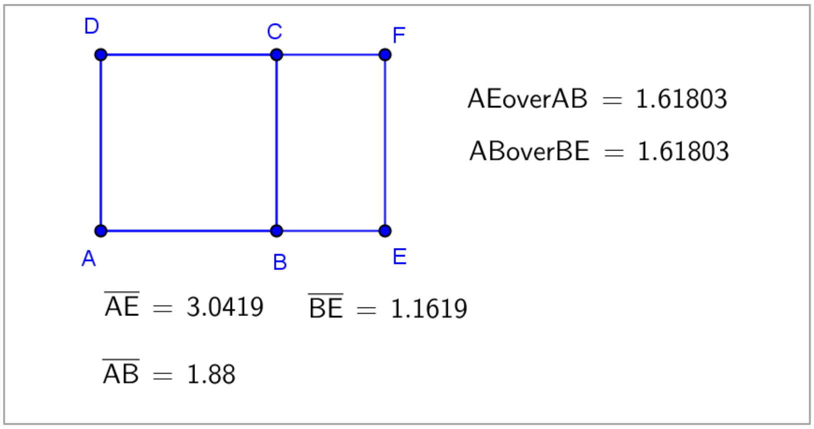


Figure 9: If rectangle AEFD is a golden rectangle, then BCFE is a golden rectangle

The regular polygon GG tool is very convenient because it allows constructing effortlessly squares. We can now use it again to construct square CFHG to generate another golden rectangle, BEHG (Fig. 10). We can continue this process to generate additional golden rectangles using the shorter side of the last rectangle as the side of a new square. This sequence is known as a golden sequence of rectangles.

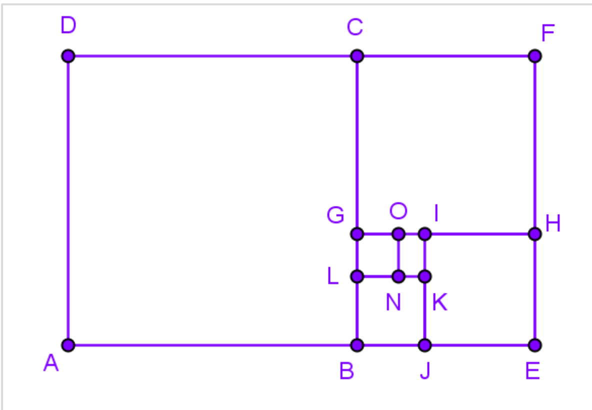


Figure 10: A golden sequence of rectangles (ADFE, BCFE, BEHG, BGIJ, GIKL, and IKNO)

5. Construction of the golden triangle

There are several geometric figures that exhibit the golden section. The next task is to consider whether the golden section is embedded in triangles. To this end, consider an isosceles triangle with a vertex angle measuring 36° and each base angle measuring 72°, as shown in figure 11a. To illustrate the versatility of GG, this time we will illustrate how to construct this triangle using transformations. The procedure is as follows:
Construct segment $\overline{AB}$, then rotate point B an angle of 72° around point A (Fig. 11a). The image of point B under this transformation is point B'. Second, determine point A', the image of point A, by rotating point A an angle of -72° around point B. Next, construct rays $\overline{AB'}$ and $\overline{BA'}$ that intersect at point C. Finally, hide rays $\overline{AB'}$ and $\overline{BA'}$, points A' and B', and the angle measures and construct segments $\overline{AC}$ and $\overline{BC}$, (Fig. 11b).

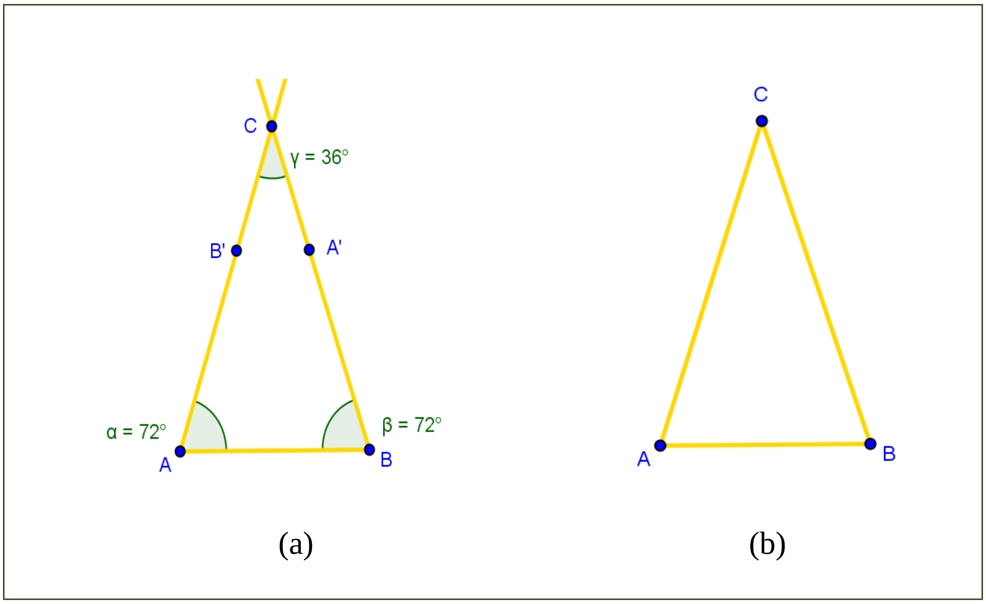


Figure 11: The construction of a 72°-72°-36° isosceles triangle using transformations

As we did with the golden rectangle, we can now ask students what property the 72°-72°-36° isosceles triangle has to have been bestowed the title golden triangle. Some of them may conjecture that the ratio between a leg and its base is the golden section. They can then use GG to compute $\frac{AC}{AB}$. As shown in figure 12a, this ratio seems to be 1.61803 …, the golden section. Further dragging of point B confirms our conjecture (Fig. 12b).

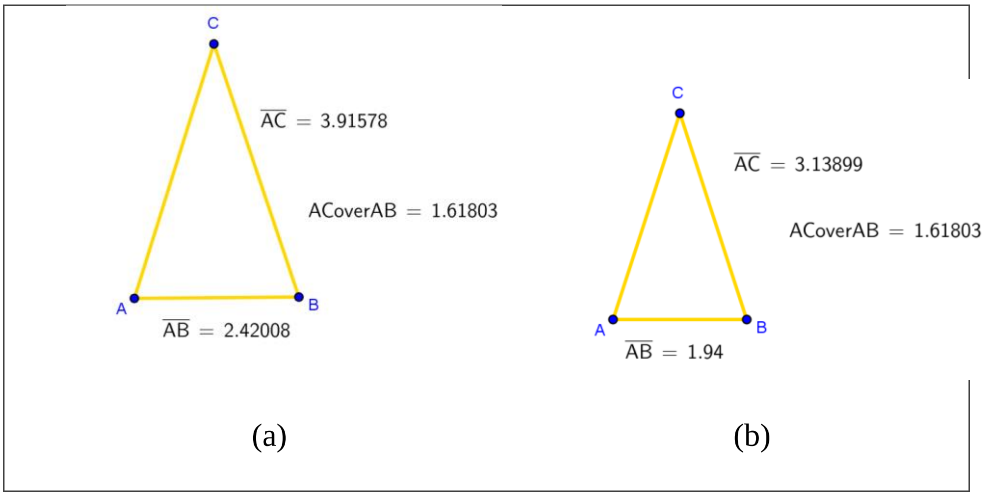


Figure 12: GG verification that isosceles ΔABC is a golden triangle

A golden rectangle has embedded a sequence of golden rectangles. Inquisitive learners may wonder whether this is the case with a golden triangle. In other words, does a golden triangle also have an embedded sequence of golden triangles? The answer is a categorical "Yes!" Constructing the angle bisector of any of the congruent angles of a 72°-72°-36° golden triangle produces another golden triangle (Fig. 13).

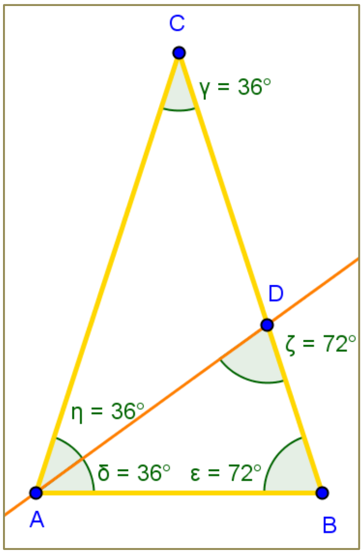


Figure 13: ΔABD is another 72°-72°-36° golden isosceles triangle

We can now justify mathematically that ΔABC, and hence ΔABD, is a golden triangle. To begin, notice that ΔABC ~ ΔBDA by the AA similarity criterion for triangles. Thus, $\frac{CB}{AD} = \frac{BA}{DB}$ because corresponding sides of similar triangles are proportional. Using the fact that BA = DA = CD we have $\frac{CB}{BA} = \frac{CD}{DB}$. That is, $\frac{CB}{CD}$ = φ and, hence, $\frac{CB}{BA} = \frac{CB}{CD} = \varphi$.

We can continue creating 72°-72°-36° golden isosceles triangles by constructing the angle bisector of appropriate 72° angles, as shown in figure 14. This generates a sequence of golden isosceles triangles.

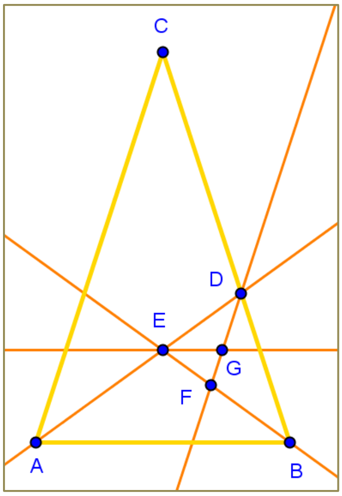


Figure 14: A sequence of golden isosceles triangles (ΔCAB, ΔABD, ΔBDE, ΔDEF, and ΔEFG)

6. Approximated construction of the golden spiral

The golden spiral is a logarithmic spiral whose growth factor is φ, the golden section. We can use the geometric configuration involving the sequence of golden rectangles constructed previously to get a spiral that approximates the golden spiral. To do so, use the GG tool "circular arc with center between two points" to construct quarter-circle arcs in each of the successive squares, as shown in figure 15.

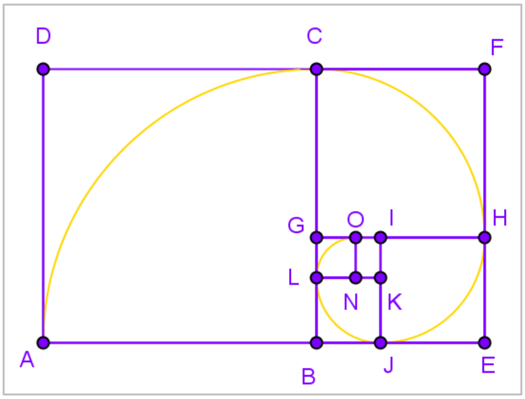


Figure 15: An approximated golden spiral using golden rectangles

We can also use the sequence of golden isosceles triangles to construct an approximation to the golden spiral using the following procedure: With center at D, construct a circular arc that goes through points C and A (Fig. 16). Repeat the process and construct circular arcs with centers E, F, G, and H as indicated in figure 16.

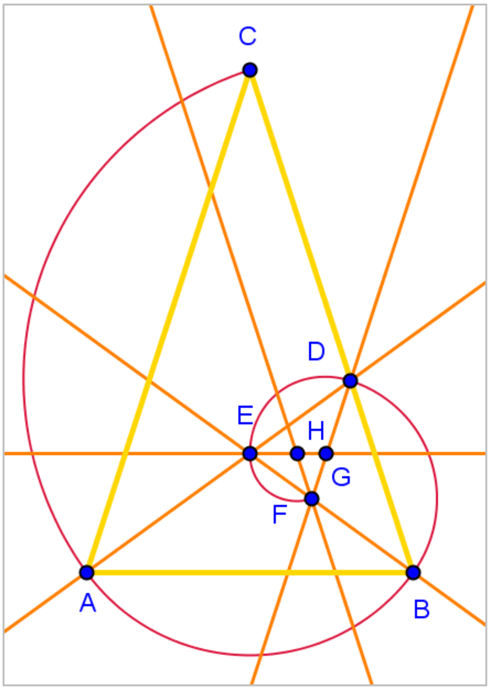


Figure 16: An approximated golden spiral using golden isosceles triangles

7. Construction of the golden pentagon and pentagram

Our last construction involves the construction of a golden pentagon. First, construct a regular pentagon using GG. Remember that a regular polygon is one in which all sides and all angles are congruent. As mentioned earlier, GG has a tool that allows us to "construct" a regular polygon with a specified number of sides.

Once we have constructed a regular pentagon using the "Regular Polygon" tool, we construct its diagonals and its circumcircle (Fig. 17). We can now explore properties of the regular pentagon related to the golden section. We invite the reader to take measurements of objects so you can discover the multiple appearances of the golden section in a regular pentagon. Since our objective is to explore the properties using GG, most of the proofs will be omitted, since most of the proofs follow the same reasoning that we have used in the previous discussions.

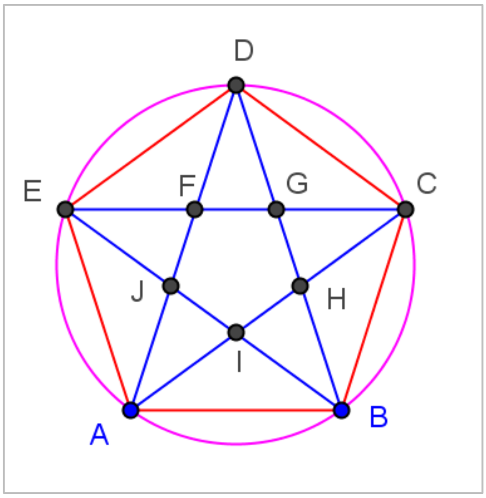


Figure 17: A regular pentagon and its diagonals

Notice that the diagonals of a regular pentagon determine a five-pointed star, which is called a pentagram (Fig. 18). It is worth mentioning that the pentagram was used as a religious symbol by the Babylonians and the Pythagoreans in ancient Greece.

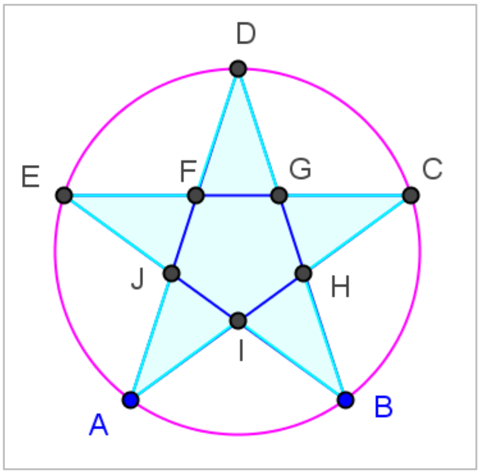


Figure 18: The pentagram

Let's now explore some of the golden properties of a regular pentagon and pentagram (Fig. 17 and 18). To begin, we notice that there are five golden triangles each having as base one of the sides of the pentagon: ΔABD, ΔBCE, ΔCDA, ΔDEB, and ΔEAC. To show that ΔABD is a golden triangle, we reason as follows: First *m*(∠BCD) = *m*(∠AED) = $\frac{3\pi}{5}$ = 108° and *m*(∠CDB) =(∠CBD) = 36° since ΔBCD is isosceles. Similarly, ∠ADE = 36° and, hence, (∠ADB) = 108° – 2(36°) = 36°. The fact that ΔBCD ≅ ΔAED (by the side angle side congruence criterion) implies that $\overline{AD} \cong \overline{BD}$. Therefore, *m*(∠DAB) ≅ (∠DBA) = $\frac{180^0-36^0}{2}$ = =72°. Thus, ΔABD is a golden triangle. Similarly, we can prove that the other four triangles are isosceles golden triangles. Triangles are: ΔGDF, ΔFEJ, ΔJAI, ΔIBH, and ΔHCG. Can the reader find more examples of golden triangles in figure 17?

To continue our search for the golden section embedded in the regular pentagon and pentagram, we can use GG to make several conjectures:

1) The ratio of the length of the diagonal to the length of one of its sides is the golden section (Fig. 19). This property shows that isosceles triangles ΔABC, ΔBCD, ΔCDE, ΔDEA, and ΔEAB are golden triangles. In other words, 36^0 – 36^0 – 108^0 isosceles triangles are also golden triangles, and this is another interesting discovery.
2) The diagonals of the regular pentagon cut each other into the golden section. (Fig. 20). This property reveals that isosceles triangles ΔBDJ, ΔCEI, ΔDAH, ΔEBG, and ΔACF are golden triangles.
3) The diagonals of a regular pentagon are divided into the golden section at two different points (Fig. 20).
4) The ratio of a side of a pentagon (e.g.,$\overline{DC}$) and the external side of its associated pentagram (e.g.,$\overline{DG}$) is the golden section (Fig. 21). This property shows that isosceles triangles ΔCDG, ΔDEF, ΔEAJ, ΔABI, and ΔBCH are golden triangles as well.
5) The ratio of the sides of the two regular pentagons is the square of the golden section (≈ 2.618033989) (Fig. 22).
6) The ratio of the radio of the inscribed circle of a regular pentagon and the radius of its circumscribed circle is half of the golden section ($\frac{\sqrt{5}+1}{4}$ ≈ 0.809016994) (Fig. 23)

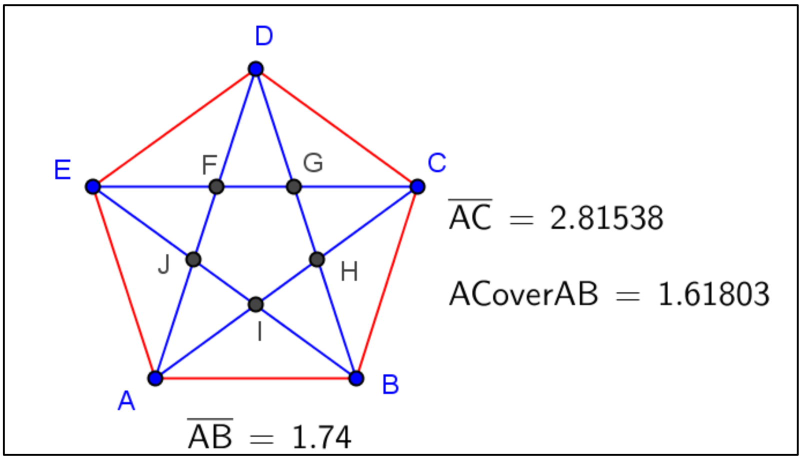


Figure 19: The ratio between AC and AB is φ

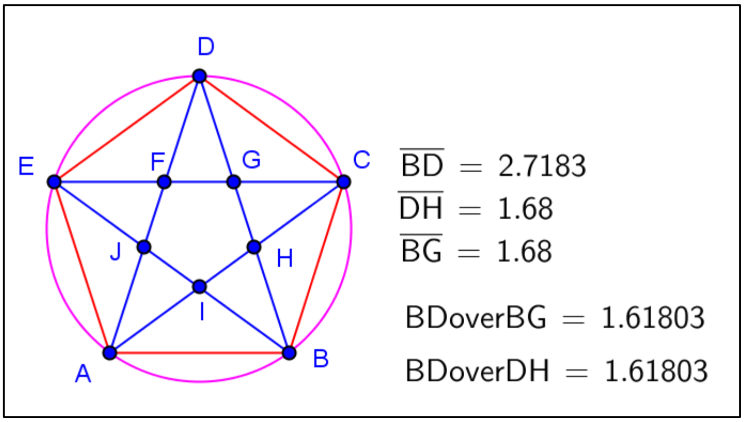


Figure 20: BD/BG = φ and BD/DH = φ

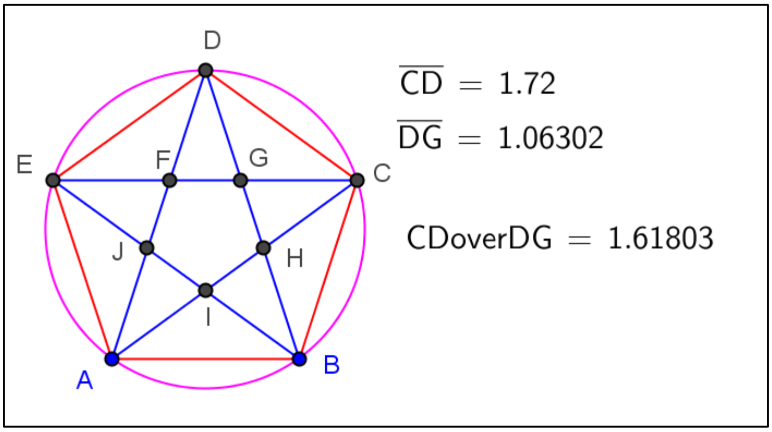


Figure 21: The ratio between CD and DG is φ

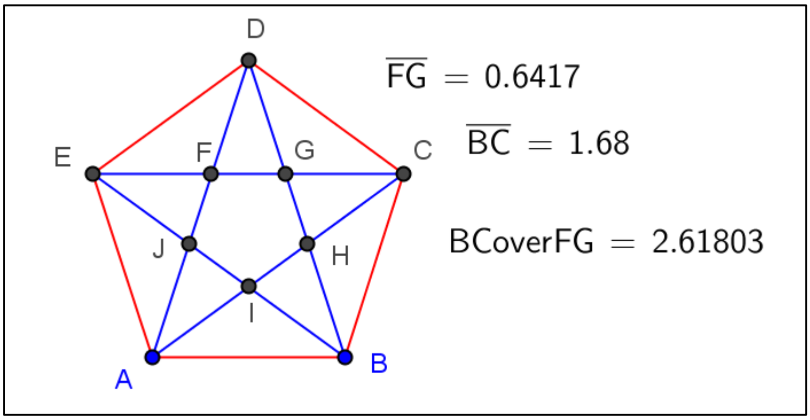


Figure 22: The ratio between BC and FG is φ^2

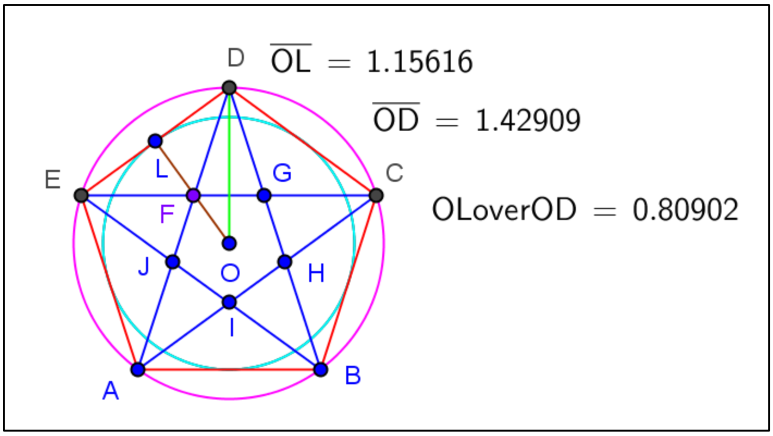


Figure 23: The ratio between the inradius and circumradius is $\varphi/2$

By now the inquisitive reader should not be surprised of discovering more examples of the golden section in a regular pentagon. In particular can you notice more isosceles golden triangles?

8. Concluding remarks

In this article we have used GG, very powerful dynamic geometry software, to illustrate how students can explore some of the appearances of the golden section in some basic figures of the geometric kingdom: the rectangle, the triangle, and the pentagon. The reader interested in learning more about the history of the golden section and its pervasive appearance not only in mathematics but also in nature and other human affairs such as art, including literature, sculpture, and architecture may consult Haubourdin (2011), Herz-Fischer (1987), Huntley (1970), Livio (2003), Olsen (2006) and Schneider (1995). We close with a quote by Kepler that reflects the ubiquitousness and beauty of the golden section: the golden ratio is a priceless jewel!

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