Infinite products and infinite sums: visualizing with the dynamic system GeoGebra

Francisco Regis Vieira Alves
Instituto Federal de Educação, Ciência e Tecnologia do
Estado do Ceará – IFCE. Brazil
fregis@ifce.edu.br

ABSTRACT: *In present article, we discuss some mathematical properties and the qualitative characters related to the notions of infinite products and infinite sums of real and complex numbers. We want to show that with the help of the dynamic system Geogebra, we can obtain and analyze the graphic-geometric behavior related to several elements. Some of these elements indicated in this paper hold great significance in the teaching context of Real Analysis and Complex Analysis. In this way, we indicate a heuristic transmission for these topics.*

KEYWORDS: Infinite products, Infinite sums, Visualization, Geogebra.

1 Infinite product and some preliminary definitions

We know that the question of convergence and divergence of infinite products can be studied formally by the infinite sums. We start our discussion mentioning that an infinite product of real (or complex) numbers is denoted by $\prod_{n=1}^{\infty} b_n$, where $b_n \in IR$ or $b_n \in C$ and not null. These elements are called by terms of the product.

We know that $\prod_{k=1}^{\infty} b_k$ is said convergent when exists a number $u \neq 0$ such that $u_n = \prod_{k=1}^{n} b_k$ converge to u. In this case, we say that the infinite product is u. On the other hand, this product is said divergent, when it is not convergent. In particular, when we have the following limit $\lim_{n \to +\infty} u_n = 0$. Traditionally, we declare that $\prod_{k=1}^{\infty} b_k$ diverges to zero. In the specialized literature, this trivial case is dismissed. Thus, we will discuss some particular and related examples to this formal mathematical definition.

In the figure 1, we show the numerical behavior of some products and sums of real numbers. In fact, we consider the infinite products $\prod_{n=1}^{\infty}\left(1-\frac{1}{n}\right)$, $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)$ and the partial sums $\sum_{n=1}^{n}\frac{1}{n}$ of the harmonic series. We visualize that the partial sums are increasing, for the some values $n \in IN$. This fact indicates that these partial sums are not limited. Thus, we understand a divergence of the harmonic series. However, when we refer to the case of the infinite products, we can extract the similar information, originated by the qualitative behavior in the same frame. Indeed, we conclude that $\prod_{n=1}^{\infty}\left(1-\frac{1}{n}\right)\to 0$ and $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)\to\frac{1}{2}$.

Here, we illustrate these formal definitions given at the beginning. We also recorded the case of $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ and, by considering the partial product $\prod_{n=1}^{K}\left(1+\frac{1}{n^2}\right)$. We too infer that $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)\to 1.82$. (fig. 1).

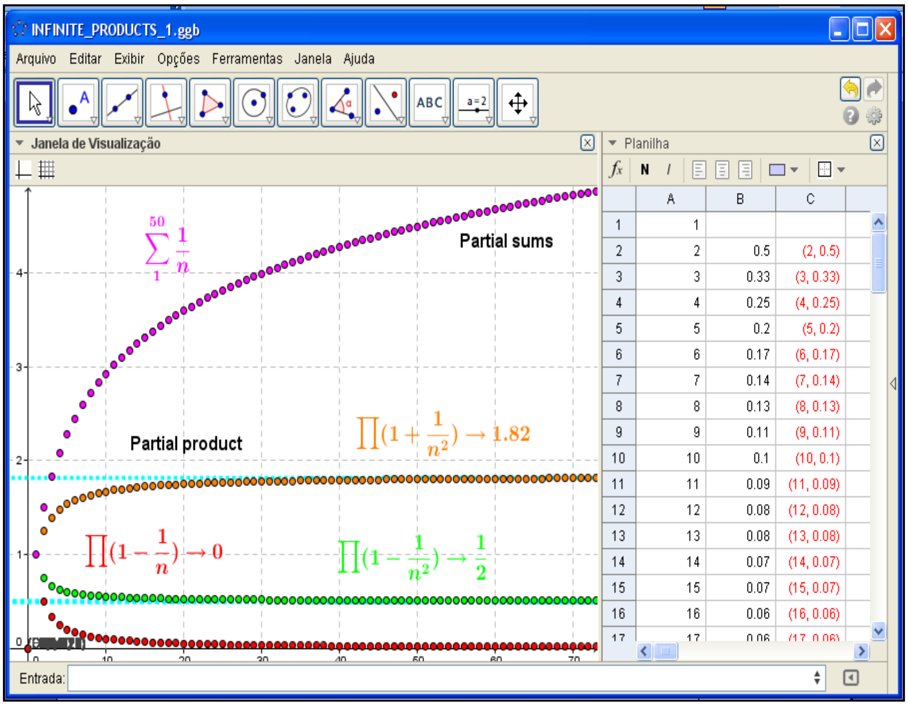

Figure 1. The dynamical system Geogebra provides the numerical and graphical behavior for the infinite products and infinite series

We extracted from the figure 1 certain properties that allows us to conjectures about the graphic-geometric behavior. However, in other to reach a definite answer related to these conjectures, we need a formal and logical model. With this concern, in the next section, we will discuss some well-known theorems in the literature.

2 Theorems and some properties

Now, we discuss some classical theorems related to the notions of infinite products and series. We will highlight that some of these formal arguments can be interpreted by the software and also indicate the important implications for the behavior of real (and complex) series (and infinite products). Thus, we enunciate our first theorem.

Theorem 1: The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges, with $a_n > 0$ if, only if, the infinite sum $\sum_{n=1}^{\infty} a_n$ converges.

The following inequality $1+x \le e^x$ (*), for all $x > 0$ can be understood by the graphical behavior of the two functions $f(x) = x+1$ and $g(x) = e^x$. The partial product $u_n = \prod_{k=1}^{n}(1+a_k)$ permit conclude that $u_n = \prod_{k=1}^{n}(1+a_k) \overset{(*)}{\le} e^{\sum_{k=1}^{n} a_k} = e^{s_n}$. Moreover, we observe even that $s_n = \sum_{k=1}^{n} a_k \le \prod_{k=1}^{n}(1+a_k)$ which é immediate consequence of $a_n > 0$. Finally, we obtain that the following inequalities $s_n \le u_n \le e^{s_n}$ holds for $a_n > 0$ and $n \ge 1$. From this inequality, we establish that s_n is limited if, only if u_n manifest the same properties. This proves the theorem 1.

We can consider too the expression $\prod_{n=1}^{\infty}(1+a_n)$ and describe a notion of absolute convergence. This properties occurs when we know that the series $\prod_{n=1}^{\infty}(1+|a_n|) < \infty$ is convergent (en virtue theorem 1 $\sum_{n=1}^{\infty}|a_n| < \infty$). In fact, we note that $\prod_{n=1}^{\infty}(1+a_n) \le \prod_{n=1}^{\infty}(1+|a_n|)$ and $a_n \ne 0$, $\forall n \in IN$. So, the convergence of the $\prod_{n=1}^{\infty}(1+|a_n|)$ implies that $\prod_{n=1}^{\infty}(1+a_n)$ converge too. From these formal arguments, we can conclude our second theorem.

Theorem 2: The infinite product $\prod_{n=1}^{\infty}(1-a_n)$ converges, with $a_n \ge 0$ if, only if, the infinite sum $\sum_{n=1}^{\infty} a_n$ converges.

$\Leftarrow$ In this case, we observe that $(1-a_n) \leq 1+|a_{\tilde{n}}|$ for $n \in IN$ (**). Now we emphasize if $\sum_{n=1}^{\infty} a_n$ converge, then $\sum_{n=1}^{\infty} a_n \overset{hip.}{=} \sum_{n=1}^{\infty} |a_n|$ converge too. By the theorem 1 and (*), we obtain the conclusion. In Tauvel (2006) we find the other verification. Let us consider the following examples: $\prod_{n=2}^{\infty}(1-\frac{2}{n(n+1)})$ and $\prod_{n=1}^{\infty}(1+\frac{(-1)^{n+1}}{n})$. Through specific algebraic manipulations, we determine, in the first case $p_m = \prod_{n=2}^{m}(1-\frac{2}{n(n+1)}) = \prod_{n=2}^{m}\left(\frac{(n-1)(n+2)}{n(n+1)}\right) =$

$$= \prod_{n=2}^{m}(n-1)\cdot\prod_{n=2}^{m}(n+2)\cdot\prod_{n=2}^{m}\frac{1}{n}\cdot\prod_{n=2}^{m}\frac{1}{n+1} = \prod_{n=1}^{m-1}n\cdot\prod_{n=4}^{m+2}n\cdot\prod_{n=2}^{m}\frac{1}{n}\cdot\prod_{n=3}^{m+1}\frac{1}{n} =$$

$$= \frac{1\cdots m-1 \;\cdot\; 4\cdots m+2}{2\cdots m \;\cdot\; 3\cdots m+1} = \frac{1\cdot(m+2)}{m\cdot 3} = \frac{1}{3}\left(1+\frac{2}{m}\right) \overset{m\to+\infty}{\to} \frac{1}{3} \therefore \lim_{m\to\infty} p_m = \prod_{n=2}^{\infty}(1-\frac{2}{n(n+1)}) = \frac{1}{3}$$

In the second case, we observe that $\prod_{n=1}^{\infty}(1+\frac{(-1)^{n+1}}{n}) = \prod_{n=1}^{\infty}u_n \therefore u_n = \left(1+\frac{(-1)^{n+1}}{n}\right)$. From this expression, we can conclude the following properties: $u_{2n} = \left(1+\frac{(-1)^{2n+1}}{2n}\right) = \left(\frac{2n-1}{2n}\right)$,

$u_{2n-1} = \left(1+\frac{(-1)^{2n-1+1}}{2n-1}\right) = \left(\frac{2n}{2n-1}\right) = \frac{2}{2-1/n} \overset{n\to\infty}{\to} 1 \therefore \forall n \geq 1$ we get still that $u_{2n-1}\cdot u_{2n} = 1$.

In this moment, we note that $p_{2n} = \prod_{n=1}^{2n}u_n = (u_1\cdot u_2\cdots u_{2n-1}\cdot u_{2n}) = 1$. Moreover, we find $p_{2n+1} = \prod_{n=1}^{2n+1}u_n = p_{2n}\cdot u_{2n+1} = 1+\frac{1}{2n+1} \overset{n\to+\infty}{\to} 1$. In the figure 2, we visualize the behavior of convergence related to the both infinite products. We also indicate, in the case of $\prod_{n=1}^{\infty}(1+\frac{(-1)^{n+1}}{n})$, the existence of the two subsequences. The two straights indicate the values of the convergence of each infinite products $y = 1/3$ and $y = 1$.

We also observe that only the latest arithmetical properties do not promote a feeling about the graphic-geometric behavior from these problems. Indeed, from the *spreadsheet window* we follow the numerical values involved in the elements showed in the figure below.

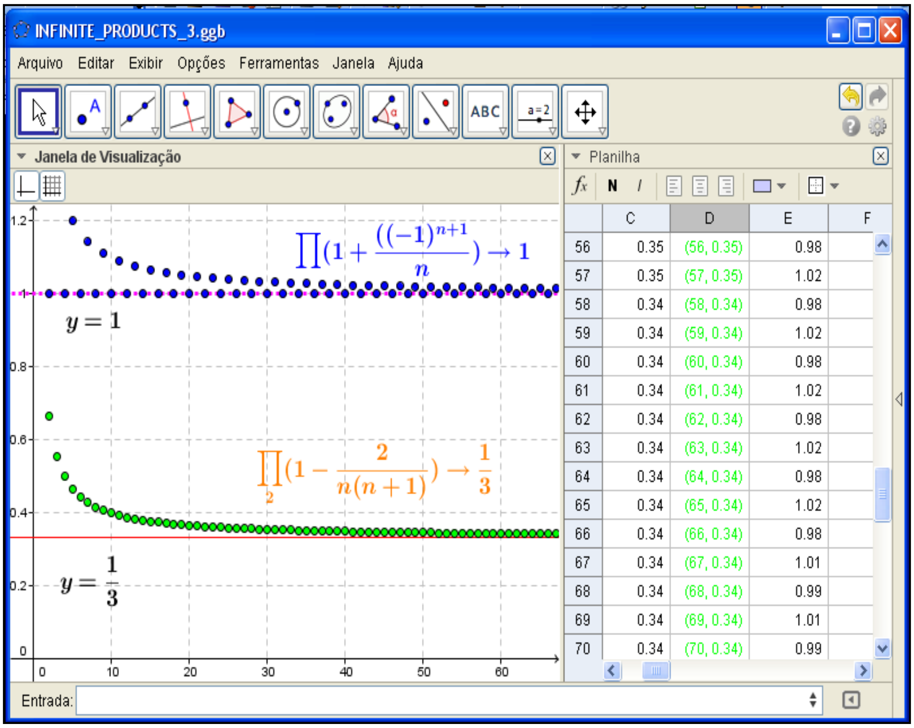

Figure 2. Visualization of the graphic behavior of the infinite product

In fact, the first two theorems admit a generalization in the case where we deal with a complex variable. In this sense, we mention the following theorem. Its demonstration can be found in Shakarchi (2000, p. 235).

Theorem 3: Consider the infinite product $\prod_{n\geq 1}^{\infty}(1+z_n)$, where $\lim_{n\to\infty} z_n = 0$, $z_n \neq -1$ for all $n \geq 1$. The infinite product converge if only if $\sum_{n\geq 1}^{\infty} z_n$. In this case, we can explore the geometric meaning related to the some formal operations defined in the Complex Analysis – CA.

Indeed, in the figure 3, we can analyze the behavior of the particular geometrical series $\sum_{n\geq 1}^{\infty} z^n$ that converge, if only if $|z|<1$. In figure 3 we indicate the behavior of the partials sums indicated here by $\sum_{n\geq 1}^{40} z^n$.

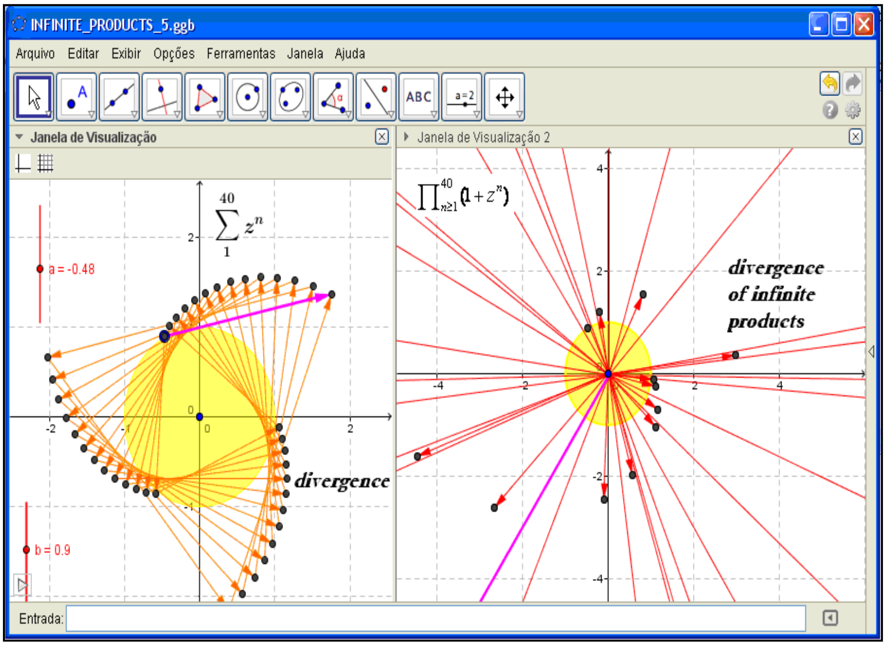

Figure 3. Visualizing the behavior of divergence in the complex variable case

The differences here deserve a careful attention. In fact, on the left side, we see the vector sums, while in the right side; we perceive a configuration like several rays emanating from the origin of the *C*-plane. The final result is a pink vector that we perceive that it has a big length. Similarly, on the left side, we note a pink vector which the length is bigger than the diameter of a disk, centered at the origin. With a moving point, we can explore the behavior of these two algebraic expressions. Indeed, we can explore some regions in the "ring of doubt" and find an unexpected behavior. In this sense, we see in the figure 4 that exist some regions in the *C*-plane which we can encounter a convergence behavior.

In the figure 4 we visualize the behavior of partial product $\prod_{n\geq1}^{40}(1+z^n)$. We emphasize the geometric interpretation of the complex product. We indicate the pink vector and investigate the radius of convergence ρ. We know that the geometric series converges for $\rho<1$, and diverge for $\rho\geq1$. We see the existence the two pink vectors. On the left side, we indicate the resultant sums of the partial series. On the right side, the pink vector indicates the result of the product $\prod_{n\geq1}^{40}(1+z^n)$. On the left side we see the behavior of the corresponding sum. Similar to what we said in the case of real numbers series so let's look at another generalization of the previous case.

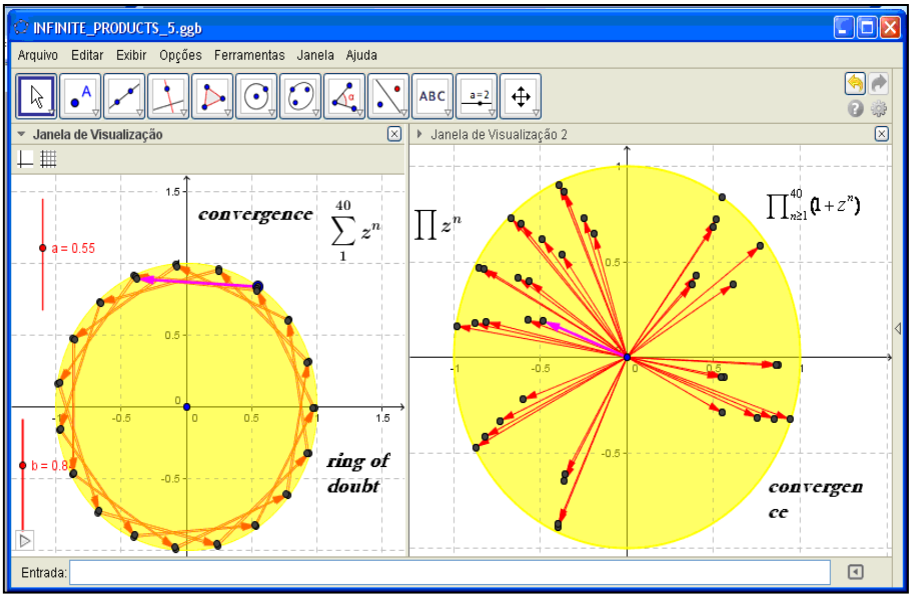

Figure 4. Visualizing the behavior of convergence/divergence in the complex variable

Theorem 4: The infinite product $\prod_{n\geq 1}^{+\infty} z^n$ converge to the number $p \neq 0$ if, only if, for all $\varepsilon > 0$, may exist $n_0 \geq 1$ such that $m > n \geq n_0$ we have $\left|\prod_{j=n}^{m} z^n - 1\right| < \varepsilon$. By the theorem 4, we can reduce the problem of infinite products by the convergence of infinite series. In fact, based on this theorem, we state the following corollary which discussed by some authors (SHAKARCHI, 2000).

Corollary: The infinite product $\prod_{n\geq 1}^{+\infty}(1+z^n)$ converge, where $\lim_{n\to\infty} z_n 0$ and $z_n \neq -1$ for all $n \geq 0$. Then, the infinite product is absolute convergent if, only if, the series $\sum_{n\geq 0}^{\infty} z_n$ is absolute convergent.

Well, as we have shown so far, we seek to provide a graph-geometric interpretation for each theorem and convergence criterion indicated here. Before turning to the problem of infinite products, we recall the notion of Wally´s integral. In fact, we know that $I_n = \int_0^{\pi/2} \sin^n(x)dx$, with the condition $n \geq 0$. From the specialized literature, we can establish that $I_{2n} = \frac{2n\ !}{2^{2n}\ \ n!} \cdot \frac{\pi}{2}$ and

$I_{2n+1} = \dfrac{2^{2n}\,(n!)^2}{(2n+1)!}$. In fact, we easily get that $I_0 = \dfrac{\pi}{2}$, $\mathrm{I}_1 = 1$, $\mathrm{I}_2 = \int_0^{\pi/2} sen^2(x)dx = \dfrac{\pi}{4}$.

Moreover, we obtain two fundamental equalities $I_n = \dfrac{n-1}{n} \cdot I_{n-2}$ and $I_{n+1} = \dfrac{n}{n+1} I_{n-1}$ $\forall n \geq 2$. From these relations, we reach the following property $0 \leq \cdots \leq I_{n+1} \leq I_n \leq I_{n-1}$ and obtain $\lim\limits_{n\to+\infty} I_n = 0$.

The sequence $(I_n)_{n\in IN}$ can be interpreted by the area's contribution under the graph of the function $f_n(x) = \sin^n(x)$. In the figure 5 we observe the blue points (on the right side). We observe the basic properties $I_n = \int_0^{\pi/2} \sin^{n-1}(x)\sin(x)dx = \left[-\cos(x)\sin^{n-1}(x)\right]_0^{\pi/2} + \int_0^{\pi/2} \sin^{n-1}(x)\sin(x)dx \therefore$

$$I_n = (n-1)\int_0^{\pi/2}(1-\sin^2(x))\sin^{n-2}(x)dx = (n-1)\left[\int_0^{\pi/2}\sin^{n-2}(x)dx - \int_0^{\pi/2}\sin^n(x)dx\right]$$

$$= (n-1)\,(I_{n-2} - I_n) \;\therefore I_n = (n-1)\,(I_{n-2} - I_n)\;, I_n = \frac{n-1}{n} I_{n-2} \;\forall \mathrm{n} \geq 2$$

From this equality, we infer that $I_{n+1} = \dfrac{n}{n+1} I_{n-1} \therefore I_n \cdot I_{n+1} = I_n I_{n+1} =$ $= \left(\dfrac{n-1}{n} I_{n-2}\right)\left(\dfrac{n}{n+1} I_{n-1}\right) \leftrightarrow (n+1)I_n I_{n+1} = (n-1)I_{n-2}I_{n-1}$. Now, we can use an inductive reasoning, in virtue that: $(n+1)I_n I_{n+1} = (n-1)I_{n-2}I_{n-1} =$

$$= (n-3)I_{n-4}I_{n-3} = (n-5)I_{n-6}I_{n-5} = \cdots = I_0 I_1 = \pi/2 \therefore (n+1)I_n I_{n+1} = \frac{\pi}{2}.$$

Finally, we conjecture the existence the limit indicated by $\lim\limits_{n\to\infty} I_n = L \therefore \lim\limits_{n\to\infty} I_n \cdot I_{n+1} = L \cdot L = \lim\limits_{n\to\infty} \dfrac{\pi}{2} \cdot \dfrac{1}{n+1} = 0$. We can visualize this last convergence on the figure 5 (on the right side). From these mathematical facts, we can analyze now the follow expression $\prod\limits_{n=1}^{\infty}\left(1 - \dfrac{1}{4n^2}\right) = \dfrac{2}{\pi}$.

In fact, we can write still that: $\left(1 - \dfrac{1}{4n^2}\right) = \dfrac{4n^2-1}{4n^2} = \dfrac{(2n-1)/2n}{2n/(2n+1)}$.

We consider again the equality: $I_n = \dfrac{n-1}{n} I_{n-2} \therefore I_{2n} = \dfrac{2n-1}{2n} I_{2n-2} =$

$$= \frac{2n-1}{2n}\frac{2n-3}{2n-2} I_{2n-4} = \frac{2n-1}{2n}\frac{2n-3}{2n-2}\frac{2n-5}{2n-4} I_{2n-6} = \frac{2n-1}{2n}\frac{2n-3}{2n-2}\frac{2n-5}{2n-4}\cdots\frac{3}{4}\frac{1}{2} I_0$$

From this result, we can use the equality $I_{2n} = \frac{(2n)!}{2^{2n}(n!)^2}\frac{\pi}{2}$ like we have mentioned.

By a equivalent argument, we reach that $I_{2n+1} = \frac{2^{2n}\,(n!)^2}{(2n+1)!}$. But, we obtain the behavior of the two quotients $\frac{I_{2n}}{I_{2n-2}} = \frac{\frac{(2n)!}{2^{2n}(n!)^2}\frac{\pi}{2}}{\frac{(2n-2)!}{2^{2n-2}((n-1)!)^2}\frac{\pi}{2}} = \frac{(2n-1)}{2n}$, $\frac{I_{2n+1}}{I_{2n-1}} = \frac{2n}{2n+1}$. Finally, we take the limit of this last expression, and obtain:

$$\prod_{n=1}^{N}\left(1-\frac{1}{4n^2}\right) = \prod_{n=1}^{N}\frac{(2n-1)/2n}{2n/(2n+1)} =$$

$$= \frac{\frac{I_{2N}}{I_{2N-2}}}{\frac{I_{2N+1}}{I_{2N-1}}}\cdot\frac{\frac{I_{2N-2}}{I_{2N-4}}}{\frac{I_{2N-1}}{I_{2N-3}}}\cdot\frac{\frac{I_{2N-4}}{I_{2N-6}}}{\frac{I_{2N-3}}{I_{2N-5}}}\cdots\frac{\frac{I_2}{I_0}}{\frac{I_3}{I_1}} = \frac{\frac{I_{2N}}{1}}{\frac{I_{2N+1}}{1}}\cdot\frac{\frac{1}{1}\frac{1}{1}}{\frac{1}{1}\frac{1}{1}}\cdot\frac{\frac{1}{1}}{\frac{1}{1}}\cdots\frac{\frac{1}{I_0}}{\frac{1}{I_1}} = \frac{\frac{I_{2N}}{1}}{\frac{I_{2N+1}}{1}}\cdot\frac{\frac{1}{1}\frac{1}{1}}{\frac{1}{1}\frac{1}{1}}\cdot\frac{\frac{1}{1}}{\frac{1}{1}}\cdots\frac{\frac{1}{\pi/2}}{\frac{1}{1}} = \frac{2}{\pi}\frac{I_{2N}}{I_{2N+2}}$$

And since $\lim_{n\to\infty}\frac{I_{2N}}{I_{2N+1}} = 1$ we finally conclude that $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) = \frac{2}{\pi}$.

This expression has a profound meaning!

In the figure 5, we show certain elements that we have indicated. First, on the left side, we can observe that $\frac{I_{2N}}{I_{2N+1}} \to 1$ from the behavior of the green points. Furthermore, we still see the red points which represent the terms of the infinite product. On the right side, we can conjecture the numerical and geometric behavior corresponding to areas´ contributions. We still see some points defined by command **Integral [f, 0, a]**. On the top of the figure 5, we indicate some basic commands that make possible the visual analysis of the area's contribution, under the graph $f_n(x) = sen^n(x)$. We perceive that these area's contributions are decreasing.

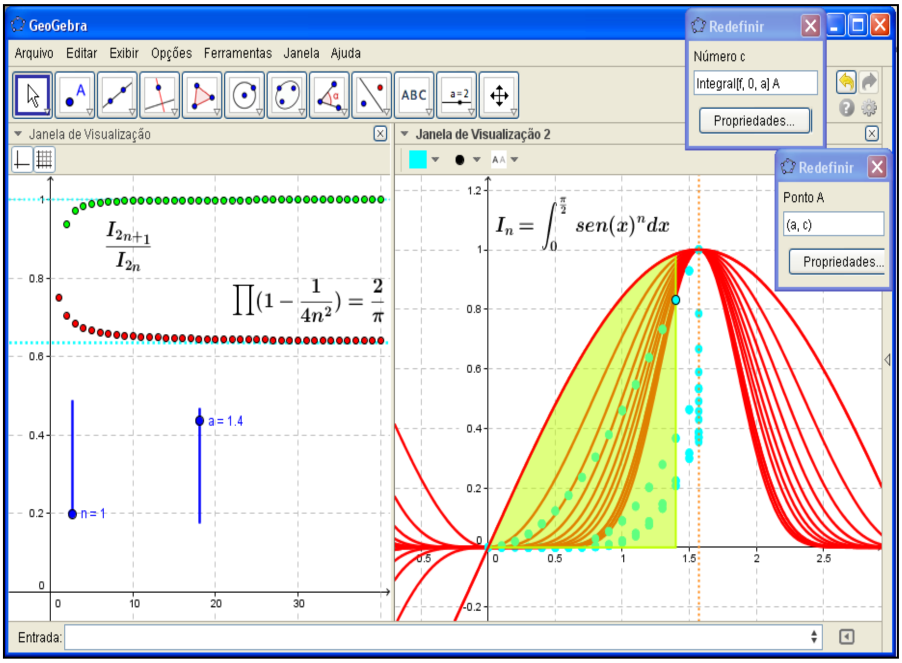

Figure 5. Visualization of the Wallis´ product and the corresponding integral

From the some History Mathematics´ books, we know the emblematic expression $\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$ (John Wallis – 1616/1703). We have written $\frac{\pi}{2}$ as a product of little fractions and $\frac{2}{\pi}$ as an infinite product.

We have to pay attention here. In fact, we could write that:

$$\frac{\pi}{2}=\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdots}.$$

In this particular manner, we could think in a terrible idea related to cancel the odds below by their doubles above. In this way, we still write: $\frac{\pi}{2}=2\cdot2\cdot4\cdot2\cdot2\cdot8\cdot8\cdot2\cdot2\cdots$ which represents an absurd, en virtue the scenario that we have commented in the previous figure.

Formally, we define the Wallis´ fractions by the form $W_k=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}$.

We now have another way of expressing the previous property by the following relation $\lim_{k\to\infty} W_k=\frac{\pi}{2}$.

We will conclude with some interesting example. In fact, we observe a complex manner for to describe the natural number $2=\prod_{n=0}^{\infty}\ 1+\ 1/2\ ^{2^n}$ and a rational number $\frac{4}{3}=\prod_{n=2}^{\infty}\left(1+\frac{2n+1}{(n^2-1)(n+1)^2}\right)$. Easily, we observe that $\frac{1}{2^{2^n}}\leq\frac{1}{2^n}$ and since $\sum_{n=2}^{\infty}\ 1/2\ ^{n}<\infty$ we obtain that $\sum_{n=2}^{\infty}1/2^{2^n}<\infty$.

Finally, en virtue the theorem 1, we conclude that $\prod_{n=0}^{\infty}\ 1+\ 1/2\ ^{2^n}$ converge, however, how to find its exact value? We can obtain an answer from analytical point of view. In fact, we write $p_k=\prod_{n=0}^{k}\ 1+\ 1/2\ ^{2^n}$ and we consider the following expression $p_k\cdot u_{k+1}=\prod_{n=0}^{k}\left(1+\left(\frac{1}{2}\right)^{2^n}\right)\left(1+\left(\frac{1}{2}\right)^{2^{k+1}}\right)=\prod_{n=0}^{k+1}\left(1+\left(\frac{1}{2}\right)^{2^n}\right)=p_{k+1}$.

We can conclude that

$$\prod_{n=0}^{\infty}\ 1+\ 1/2\ ^{2^n}\ =\lim_{k\to+\infty}\prod_{n=0}^{k}\ 1+\ 1/2\ ^{2^n}\ =\lim_{k\to+\infty}p_k=2.$$

In second case, we observe in the figure 6 that $\sum_{n=2}^{k}\frac{2n+1}{(n^2-1)(n+1)^2}<\sum_{n=2}^{k}\frac{3}{n^2}$. So, by the theorem 1, we will compare with the corresponding infinite products. Finally, we can state that $\prod_{n=2}^{\infty}\left(1+\frac{2n+1}{(n^2-1)(n+1)^2}\right)$ converge, although we do not know its numerical value. In the figure 6 we indicate $y=4/3$ and the numerical values of the partial product do not cross this straight.

On the other hand, we find that: $\prod_{n=2}^{\infty}\left(1+\frac{2n+1}{(n^2-1)(n+1)^2}\right)=\prod_{n=2}^{\infty}\left(\frac{n^3(n+2)}{(n-1)(n+1)^3}\right)$ and $p_{m+1}=\prod_{n=2}^{m+1}\left(\frac{n^3(n+2)}{(n-1)(n+1)^3}\right)$. Finally, we analyze

$$p_{m+1}=\frac{2^3(2+2)}{(2-1)(2+1)^3}\cdot\frac{^3(3+2)}{(3-1)(3+1)^3}\cdot\frac{2^3(2+2)}{(2-1)(2+1)^3}\cdots$$

$\cdots \frac{m^3(m+2)}{(m-1)(m+1)^3}\cdot\frac{(m+1)^3(m+1+2)}{(m+1-1)(m+1+1)^3}$. We can find a way to simplify this long expression. After some simplifications, we still get that $p_{m+1}=\frac{4}{3}\cdot\left(\frac{m^3+6m^2+11m+6}{m^3+6m^2+12m+8}\right)\xrightarrow{m\to+\infty}\frac{4}{3}\cdot 1=\frac{4}{3}$ in according to the figure 6.

Therefore, we can infer now that its numerical value is $4/3$.

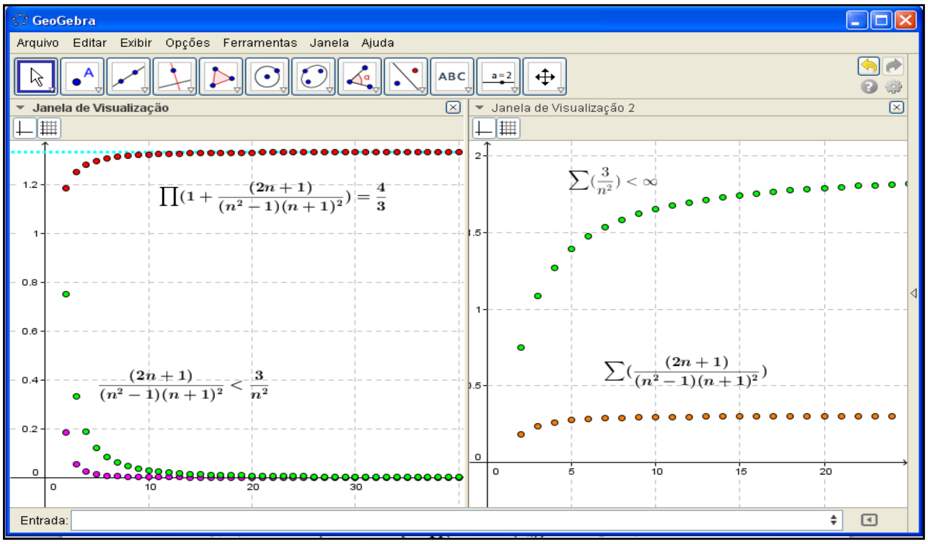

Figure 6. Visual example of application of theorem 1 with the Geogebra

Final remarks

In this work we discussed some properties related to the infinite products and infinite series. We evidenced several properties (and theorems) on the graphical-geometric frame. In the specific approach, we showed some examples that promote the visualization and the perceptual meaning of the complex concepts, among them, the notion of convergence.

Indeed, throughout the text we brought some examples of classical theorems (BOTTAZZINI, 1986; SHAKARCHI, 2000; TAUVEL, 2006) with strong intuitive appeal, in the condition that we explore the current technology. We still remember that certain formal arguments hinder the historical evolution (EDWARDS, 1979) of the intuition and the tacit reasoning.

We still must observe that the theorems and examples discussed here are not unprecedented, however, are reinterpreted with the support of technology (ALVES, 2014). Indeed, on several occasions, we realized the conceptual link between the series and infinite products, from the visual and perceptual point of view (see figures 1, 3, 4 and 5).

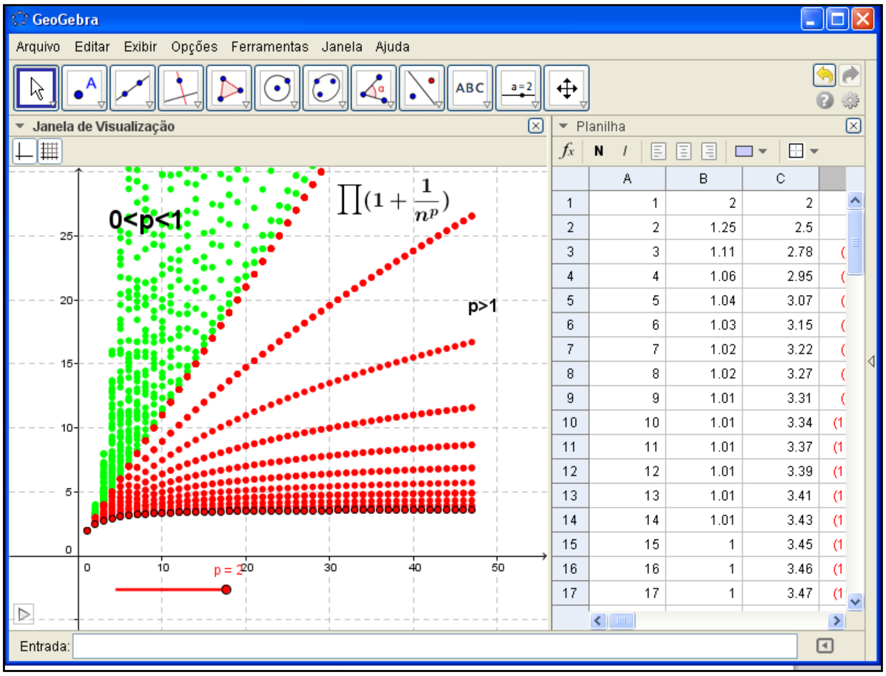

Figure 7. Visual proof of the formal properties related to the infinite products

We conclude by recording the attitude of some mathematicians that stimulated in yours academic students the imaginative ability to see theorems and formal properties (KLINE, 1953; RUSSEL, 1956). To see for example that $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^p}\right)$ converge if, only if $p>1$. And diverge in the case $0<p\leq 1$. We indicate and "see" easily this formal property in the figure 7 with Dynamic System Geogebra.

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