Investigating Variations of Varignon's Theorem Using GeoGebra

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ABSTRACT: In this paper I illustrate how learners can use GeoGebra to discover, visualize, and investigate variations of Varignon's theorem. The variations investigated include extensions to non-standard quadrilaterals (such as crossed and degenerate quadrilaterals), *n*-section points, and points that divide the sides of a quadrilateral into proportional sides.

KEYWORDS: Varignon's theorem, GeoGebra, discovery, extending, *n*-section points, crossed quadrilaterals, degenerate quadrilaterals and proof.

1. Introduction

For me, as a learner of mathematics, three of the most inspiring learning activities are to be able to discover, extend, and justify mathematical patterns and relationships. As a teacher of mathematics, I find it exciting to guide and engage my students in the process of creating new mathematics, at least new to them. The availability of Dynamic Geometry Software, such as GeoGebra (GG), facilitates and motivates students to pursue complex investigations and formulate conjectures. One of the most fruitful geometric theorems is the so called Varignon's theorem. My version of this theorem is as follows: The midpoints of the consecutive sides of a quadrilateral are the vertices of a parallelogram. This theorem is named after the French scholar and mathematician Pierre Varignon (1654-1722) who first rigorously proved it.

2. Discovering and proving Varignon's theorem

To provide students opportunities to discover Varignon's theorem, I often reformulate it as a problem:

Let E, F, G, and H be the midpoints of the consecutive sides of a quadrilateral ABCD. What type of quadrilateral is EFGH? (Fig. 1).

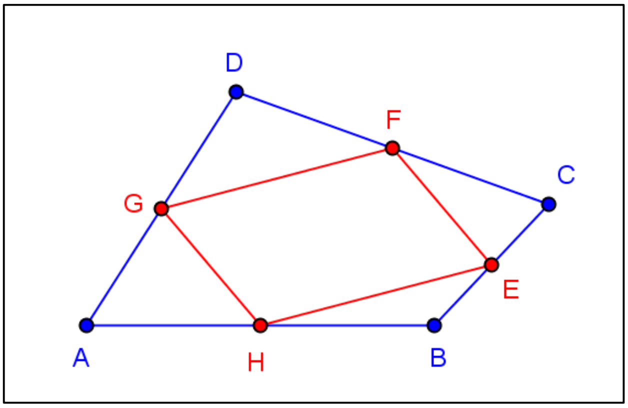
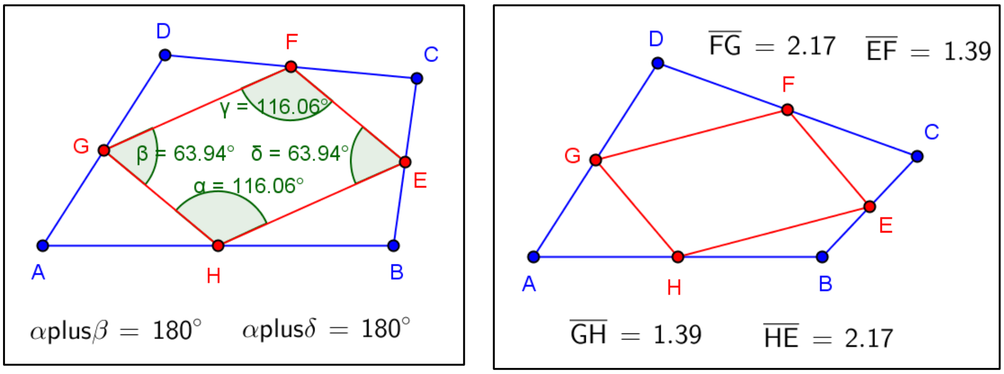


Figure 1: Varignon's Diagram

Learners can use GG to explore the problem and be delighted to discover that EFGH is a parallelogram. They can then also use the measurement capabilities of GG to test their conjecture using a variety of approaches, as indicated in figure 2. The first approach relies on the fact that two lines, or segments in the present case, are parallel if the consecutive interior angles are supplementary. Thus, $\overline{EF} \parallel \overline{GH}$ and $\overline{HE} \parallel \overline{FG}$, which implies that EFGH is a parallelogram.



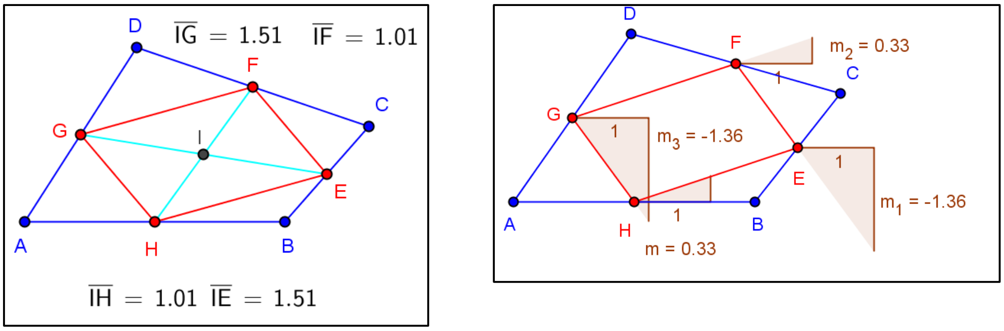


Figure 2: A diversity of strategies can be used to verify that EFGH is a parallelogram using GeoGebra

The second approach is based on another characterization of parallelograms, namely, a quadrilateral whose opposite sides are congruent is a parallelogram. Likewise, a quadrilateral is a parallelogram if and only if its diagonals bisect each other. Since the diagonals of EFGH seem to intersect each other at their respective midpoints, as shown in figure 2, we conjecture that EFGH is a parallelogram. Finally, students may also use the fact that two lines or segments are parallel if their slopes are equal or undefined. This idea is used in the fourth diagram to show that EFGH is a parallelogram because the slopes of its opposite sides are equal, implying that $\overline{EF} \parallel \overline{HG}$ and $\overline{HE} \parallel \overline{GF}$.

Once students have used GG to verify empirically that EFGH is a parallelogram, they can be asked or guided to develop a mathematical proof, if appropriate. Like most, if not all, mathematical theorems, Varignon's theorem can be proved using a variety of strategies. The following argument is simple, beautiful, and elegant:

Let E, F, G, and H be the midpoints of the consecutive sides of a quadrilateral ABCD (Fig. 3). Construct the diagonals $\overline{AC}$ and $\overline{BD}$. $\overline{EF}$ is a midsegment of ΔBCD. The midsegment theorem (A midsegment of a triangle is parallel to the third side) implies that $\overline{EF} \parallel \overline{BD}$. The same theorem applied to ΔABD leads to the conclusion that $\overline{BD} \parallel \overline{HG}$. Using the transitive property of parallelism, we conclude that $\overline{EF} \parallel \overline{HG}$. A similar argument shows that $\overline{GF} \parallel \overline{HE}$. Thus, EFGH is a parallelogram because its opposite sides are parallel.

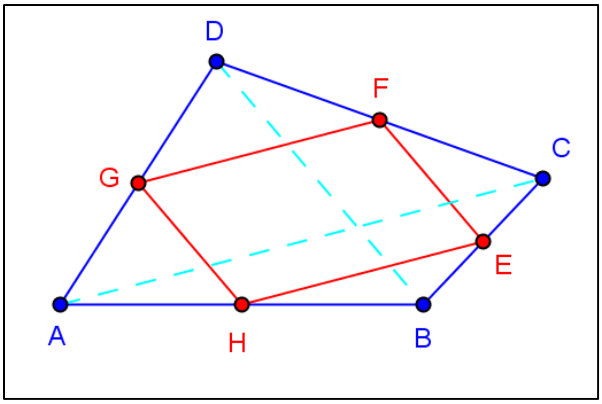
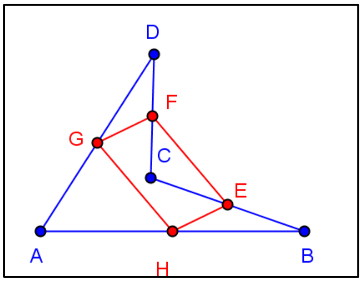


Figure 3: Diagram to prove Varignon's theorem

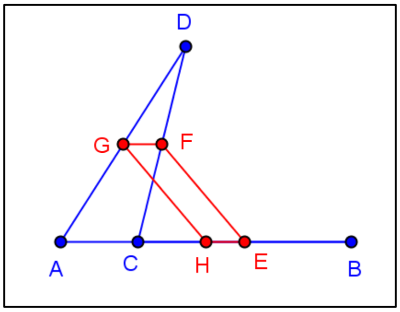
Varignon's theorem is also known as the parallelogram theorem because EFGH is a parallelogram, as the proof above shows.

3. Extending Varignon's theorem to non-standard quadrilaterals

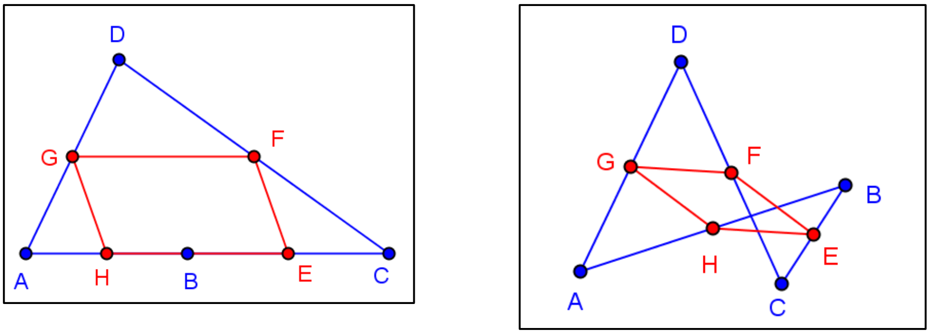
The beauty and power of a mathematical theorem lies on its capacity to be extended to different cases. By these standards, Varignon's theorem qualifies as a beautiful and powerful result in elementary geometry: it is true not only for ordinary quadrilaterals, but also for non-standard quadrilaterals such as non-convex, crossed, and degenerate quadrilaterals. GG is very helpful in visualizing the plausibility of the Varignon's theorem for these types of quadrilaterals, as figure 4 shows. The proof discussed above can be applied to these types of quadrilaterals.



Non-convex quadrilateral



Degenerate quadrilateral



Another degenerate quadrilateral

A crossed quadrilateral

Figure 4: Extending Varignon's theorem to non-standard quadrilaterals

4. Extending Varignon's theorem to other points

A natural question or problem to investigate is whether Varignon's theorem can be extended to other types of points such as trisection, five-section, or, in general, *n*-section points. As we can see from figures 5 and 6, it seems that appropriate trisection or five-section points determine parallelograms. EHIL and FGJK are parallelograms in figure 5 and $A_1B_4C_1D_4$, $A_2B_3C_2D_3$, $A_3B_2C_3D_2$, and $A_4B_1C_4D_1$ are parallelograms in figure 6.

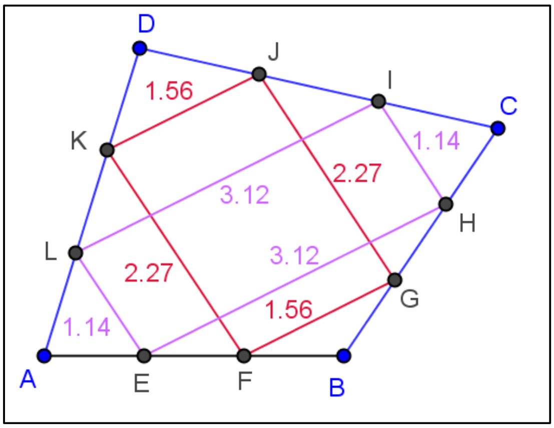


Figure 5: Parallelograms determined by appropriate trisection points of the sides of a quadrilateral

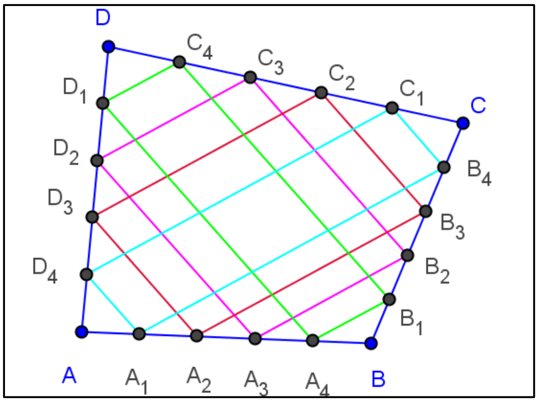


Figure 6: Parallelograms determined by appropriate five-section points of the sides of a quadrilateral

We can use Thales theorem, a generalization of the midsegment in a triangle theorem, to prove that the quadrilaterals embedded in figures 5 and 6 are parallelograms. Thales theorem can be stated as follows: A line that divides two sides of a triangle proportionally is parallel to the third side. A proof that $A_2B_3C_2D_3$ is a parallelogram follows:

Construct diagonals $\overline{AC}$ and $\overline{BD}$. Consider ΔBCD first. Because $\frac{CC_2}{C_2D} = \frac{CB_3}{B_3B}$ ($\frac{2}{3} = \frac{2}{3}$) we conclude that $\overline{C_2B_3} \parallel \overline{DB}$. Consider ΔABD next, the equality $\frac{AA_2}{A_2B} = \frac{AD_3}{D_3D}$ ($\frac{2}{3} = \frac{2}{3}$) implies that $\overline{DB} \parallel \overline{D_3A_2}$. Thus, $\overline{C_2B_3} \parallel \overline{D_3A_2}$. A similar argument shows that $\overline{D_3C_2} \parallel \overline{A_2B_3}$. We infer that $A_2B_3C_2D_3$ is a parallelogram because its opposite sides are parallel.

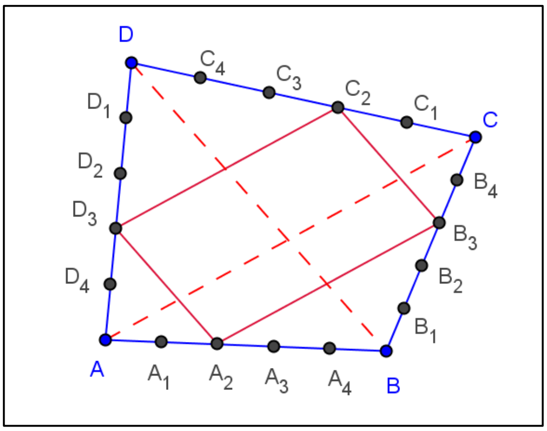


Figure 7: Diagram to prove that $A_2B_3C_2D_3$ is a parallelogram

Examining figures 5 and 6 helps up to conclude the following general result involving n-section points: Let ABCD be a quadrilateral in which each side is divided into n congruent segments. Points A_1, A_2, A_3, …, and A_{n-1} are the n-section points of side $\overline{AB}$. Points B_1, B_2, B_3, …, and B_{n-1} are the n-section points of side $\overline{BC}$. Points C_1, C_2, C_3, …, and C_{n-1} are the n-section points of side $\overline{CD}$. Points D_1, D_2, D_3, …, and D_{n-1} are the n-section points of side $\overline{DA}$. The quadrilateral $A_iB_{n-i}C_i$ D_{n-i} is a parallelogram. The previous proof can be extended to the present case.

We can now go beyond n-section points and investigate the extension of Varignon's theorem to points that divide the sides of any quadrilateral into proportional segments. Figure 8 suggests indeed that EFGH is a parallelogram, which we can prove as follows:

Since $\frac{AE}{EB} = \frac{AH}{HD}$ and $\frac{GC}{DG} = \frac{FC}{BF}$, Thales theorem implies that $\overline{EH} \parallel \overline{BD}$ and $\overline{BD} \parallel \overline{FG}$ and, as a consequence, $\overline{EH} \parallel \overline{FG}$. Similarly, $\overline{EF} \parallel \overline{HG}$. Thus, EFGH is a parallelogram because its opposite sides are parallel.

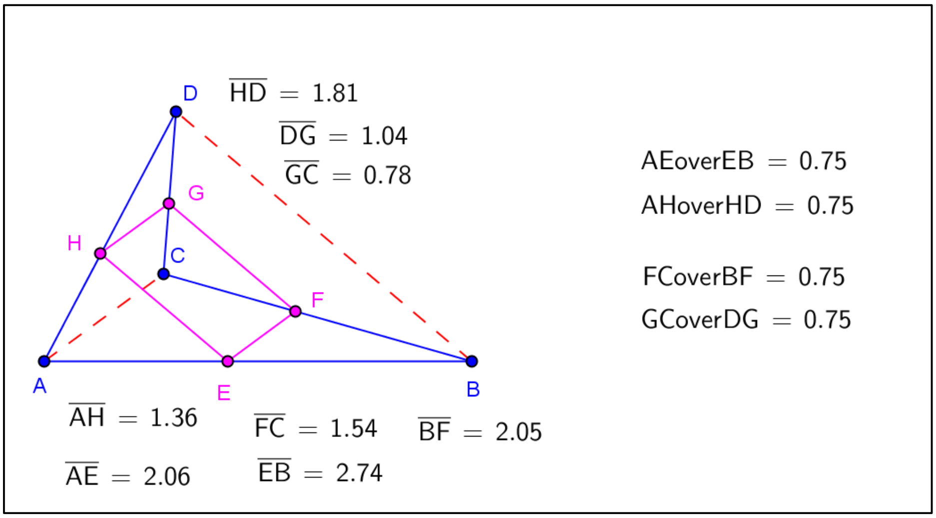


Figure 8: Points E, F, G, and H divide the sides of ABCD into proportional segments

5. Concluding remarks

Mathematicians (e.g., Polya, 1945), mathematics educators (e.g., Brown & Walter, 1990), professional organizations (e.g., NCTM, 1989, 2000), and private institutions (e.g., National Research Council, 2001) call for students to have opportunities to pose problems regularly by modifying the conditions of a given problem or theorem.

As illustrated in this paper, the use of GG facilitates tremendously the exploration or investigation of "what-if" questions.

As stated by Movshovits-Hadar (1988), most theorems (and problems) are an endless source of surprise. Varignon's theorem is certainly not an exception because it can be extended to a variety of situations, some of which were investigated in this paper with the help of GeoGebra. I invite teachers to use this problem with their students so they can experience the thrill of discovering new theorems, at least new to them. I also encourage the reader to modify other attributes of Varignon's problem to feel the joy of creating problems, conjectures, and theorems.

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