Visualizing Bezier´s curves: some applications of Dynamic System Geogebra

Francisco Regis Vieira Alves
Instituto Federal de Educação, Ciência e Tecnologia do
Estado do Ceará – IFCE. Brazil
fregis@ifce.edu.br

ABSTRACT: In this paper we discuss some properties and visual characteristics of the emblematic Bézier´s curves. Indeed, we find in the standard literature in Algebraic Geometry some briefly indications about it. Moreover, we can obtain particular algebraic examples of applications (the Bernstein polynomials $B_{n,i}(t)$). On the other hand, with the Dynamic System Geogebra - DSG, we can visualize certain situations and qualitative characteristic that are impossible to perceive and understanding without the actual technology.

KEYWORDS: Visualization, Bezier´s curves, Geogebra.

1 Introduction

In a general context, we find several methods related to seek a desired curve that passes through specified points. Related with this goal, we can observe that "a Bézier curve is confined to the convex hull of the control polygon that defines it" (BARSKY, 1985, p. 2). So, among these methods, the Bezier curve can promote a "visual adjust correspondently the points distribution". This author indicates still some graphics and a close relationship with the Bernstein's polynomials (for the cases $n=5$ and $n=6$) (see fig. 1).

Vainsencher (2009, p. 116) explains that "the Bezier´s curves can be used with certain computational and aesthetic advantages. Are provided that now, we are content with a rational curve that 'visually adjust' the graphic distribution of points". Precisely, when we considering

$$P_1=(x_1,y_1), P_2=(x_2,y_2), P_3=(x_3,y_3),\ldots,P_d=(x_d,y_d)$$

we seek a rational curve which passes through of these points. Moreover, we need that the tangents at these points containing the segments $P_1P_2, P_2P_3, P_3P_4,\ldots,P_{d-1}P_d$.

In the next section we will show some examples related to these consideration. On the other hand, en virtue to understanding this basic construction, we need to talk some information about the Bernstein's polynomials (see fig. 1). Indeed, we know the Bernstein's polynomials defined by $B_{i,n}(t)=\binom{n}{i}t^i(1-t)^{n-i}=\left(\frac{n!}{i!(n-i)!}\right)t^i(1-t)^{n-i}$, with $0\le i\le n$. (*)

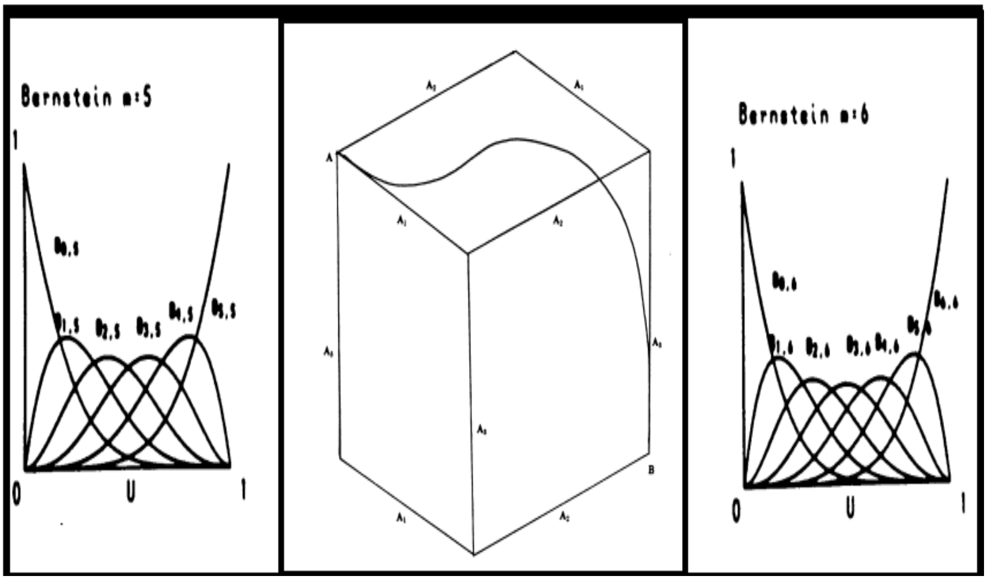

Figure 1. Visualization of a twisted curve inside a rectangular parallelepiped in three-space and the Bernstein's polynomials (BARSKY, 1985, p. 3-6)

In a general way, we define a Bézier curve as a spline curve that uses the Berstein polynomials as a basis. A Bézier curve $r(t)$ of degree n is represented by $r(t)=\sum_{i=0}^{n} b_i B_{i,n}(t)$ under condition $0\le t\le 1$. One way to computing a point of a Bezier curve, is first to evaluate the Berstein polynomials. Another, and more direct method, is the Casteljau's algorithm (see figure 3).

In the figure 1, we indicate some particular cases related to the family $\left\{B_{i,n}(t)\right\}_i^n$. In the figure 2, we exhibit a simple example provided by DSG. In fact, from the definition (*), we visualize for $0\le n\le 10$ and $0\le i\le n$. On the left side we constructed a moving point and in the right side, with the command **curve[ ]**, we see some parameterized curves. We can understand the dynamic meaning of this construction. However, before to develop other examples, we must comment a little historical context.

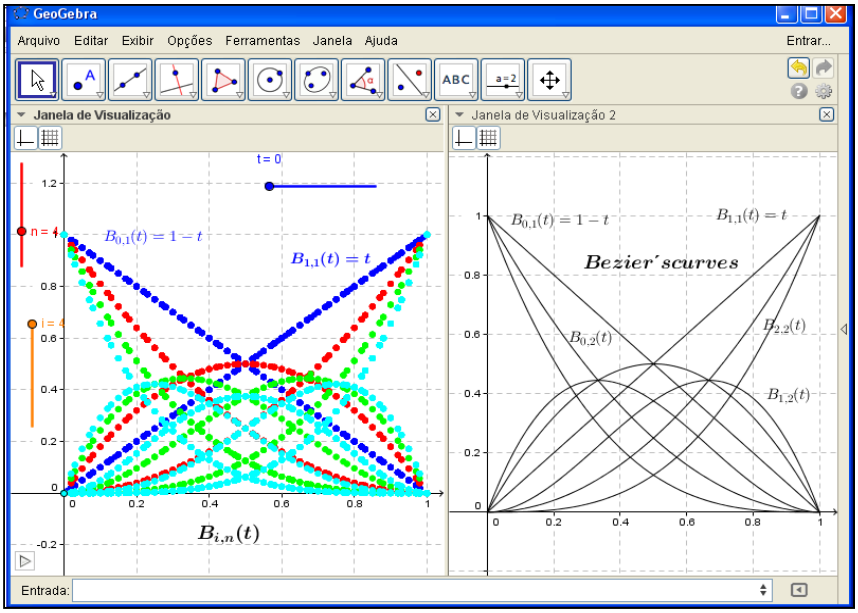

Figure 2. Visualization of Bernstein's polynomials with the DSG´s help

2 The Bezier´s curves and some properties

In the historic context, we know that Bezier´s theory curves were developed independently by P. Casteljau in 1959 and by P. Bezier in 1962. Both approaches are based on the Bernstein's polynomials. This polynomials class is known in approximation theory. Vainsencher (2009, p. 115-116) shows an application (and a little figure), although we identify the author's intention on a heuristic idea of a curve related to a transmission through drawing a graph. So closely with this heuristic idea, in the figure 3, Bertot, Guillot & Mahboubi (2010) explain that "De Casteljau algorithmic is extensively used in computer graphics for rasterizing Bézier curves".

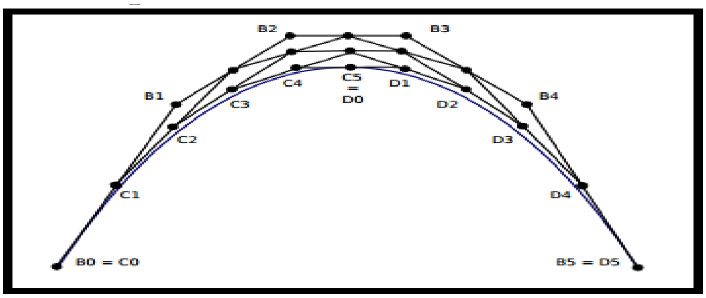

Figure 3. Bertot, Guillot & Mahboubi (2010, p. 21) illustrate De Casteljau algorithmic

A parameterization is obtained from a recursive way (VAINSENCHER, 2009, p. 116). Indeed, we can start with the polygonal determined by the $d-1$ segments. We write $\begin{cases} \sigma_1^1(t) = (1-t)P_1 + tP_2 \\ \sigma_2^1(t) = (1-t)P_2 + tP_3 \\ \dots \\ \sigma_{d-1}^1(t) = (1-t)P_{d-1} + tP_d \end{cases}$. In the next step we must substitute each consecutives pairs of polygonal by an interpolation. In our case, we take a parabola in the following way: $\begin{cases} \sigma_1^2(t) = (1-t)\sigma_1^1(t) + t\sigma_2^1(t) \\ \sigma_2^2(t) = (1-t)\sigma_2^1(t) + t\sigma_3^1(t) \\ \dots \\ \sigma_{d-2}^2(t) = (1-t)\sigma_{d-2}^1(t) + t\sigma_{d-1}^1(t) \end{cases}$. Lets consider the set of the points: $P_1 = (-1,1), P_2 = (0,0), P_3 = (-1.2,-1.2), P_4 = (2,-1.5)$. Easily, we find:

$$\begin{cases} \sigma_1^1(t) = (1-t)P_1 + tP_2 = (1-t)\cdot(-1,1) + t\cdot(0,0) = (t-1,1-t) \\ \sigma_2^1(t) = (1-t)P_2 + tP_3 = (1-t)(0,0) + t(-1.2,-1.2) = (-1.2t,-1.2t) \\ \sigma_3^1(t) = (1-t)P_3 + tP_4 = (1-t)(-1.2,-1.2) + t(2,-1.5) = (3.2t-1.2,-0.3t-1.2) \end{cases}.$$

In the next step, we compute:

$$\begin{cases} \sigma_1^2(t) = (1-t)(t-1,1-t) + t(-1.2t,-1.2t) = (-2.2t^2+2t-1,-0.2t^2-2t+1) \\ \sigma_2^2(t) = (1-t)(-1.2t,-1.2t) + t(3.2t-1.2,-0.3t-1.2) = (4.4t^2-2.4t,0.9t^2-2.4t) \end{cases}.$$

Finally, we find the following parameterized cubic curve $\sigma_1^3(t) = (1-t)\sigma_1^2(t) + t\sigma_1^2(t) = (6.6t^3-6.6t^2+3t-1,1.1t^3-0.6t^2-3t+1)$.

We still take the set of points: $Q_1 = (-1,0), Q_2 = (0,0), Q_3 = (1,2), Q_4 = (2,0)$. Following a similar procedure, we establish that: $\begin{cases} \sigma_1^1(t) = (1-t)(-1,0) + t(0,0) = (t-1,0) \\ \sigma_2^1(t) = (1-t)(0,0) + t(1,2) = (t,2t) \\ \sigma_3^1(t) = (1-t)(1,2) + t(2,0) = (1+t,2-2t) \end{cases}$. We continue by the calculations $\begin{cases} \sigma_1^2(t) = (1-t)(t-1,0) + t(t,2t) = (t,2t^2) \\ \sigma_2^2(t) = (1-t)(t,2t) + t(1+t,2-2t) = (2t,4t-4t^2) \end{cases}$ and, finally, we obtain a Bezier cubic $\sigma_1^3(t) = (1-t)(t,2t^2) + t(2t,4t-4t^2) = (t+t^2,6t^2-6t^3)$. In the figure 4, we show the dynamic construction associated to which of these recursive parameterization for four control points.

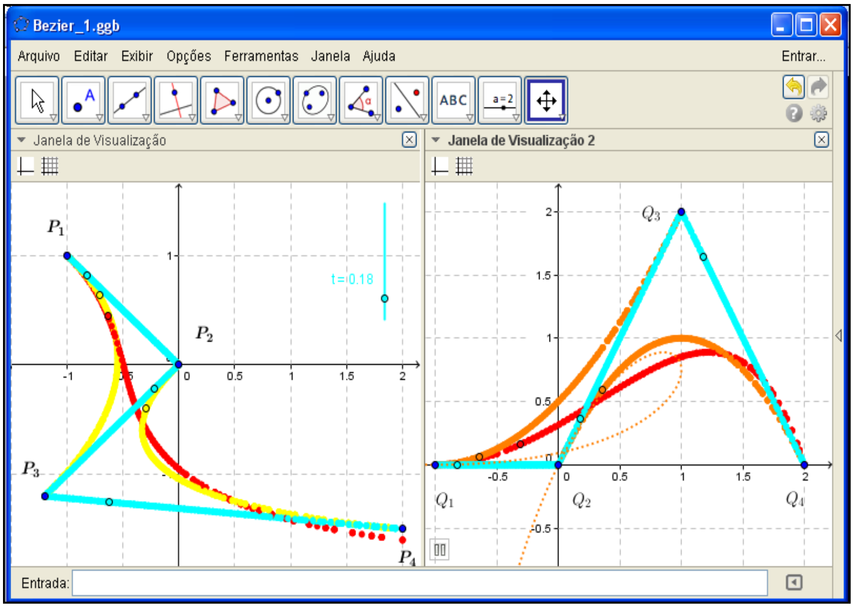

Figure 4. Visualization of some example discussed in Vainsencher (2009, p. 117).

In the specialized literature, we still find the definition correspondently a cubic trigonometric Bézier. For such, we take $\lambda, \mu \in [-1,1]$ and the parameter $t \in [0, \pi/2]$ and consider the expressions

$$\begin{cases} B_{0,3}(t,\lambda,\mu) = 1-(1+\lambda)sen(t)+\lambda sen^2(t) \\ B_{1,3}(t,\lambda,\mu) = (1+\lambda)sen(t)-(1+\lambda)sen^2(t) \\ B_{2,3}(t,\lambda,\mu) = (1+\mu)\cos(t)-(1+\mu)\cos^2(t) \\ B_{3,3}(t,\lambda,\mu) = 1-(1+\mu)\cos(t)+\mu\cos^2(t) \end{cases}.$$

This reader can analyze other properties in the work of Liu; Li & Zang (2011). Moreover, only with some basic commands of DSG, we can obtain its graphical dynamical behavior. Is easy observe that $B_{0,1}(t) \geq 0$ and $B_{1,1}(t) \geq 0$ $(0 \leq t \leq 1)$. In the figure 2 we visualize this property. In fact, by mathematical induction, we can infer that $B_{i,n}(t) \geq 0$ for all $n \in IN$. In fact, if we assume that all Bernstein polynomials of degree less than 'n' are non-negative, then by using the recursive definition of the Bernstein polynomial, we can write: $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + t \cdot B_{i-1,n-1}(t)$. (**)

3 Visualization provided by DSG and the Bernstein polynomials

In this section we will discuss some conceptual properties related to the Bernstein polynomials. En virtue its definition in (*), we can get easily that: $B_{0,1}(t)=1-t, B_{1,1}(t)=t, B_{0,2}(t)=(1-t)^2, B_{1,2}(t)=2t(1-t), B_{2,2}(t)=t^2, B_{0,3}(t)=(1-t)^3$, $B_{1,3}(t)=3t(1-t)^2, B_{2,3}(t)=3t^2(1-t), B_{3,3}(t)=t^3$, etc. In the specialized literature, we find several properties related to this notion.

In fact, from some of these properties, we can extract a formula to the Quadratic and Cubic Bezier curve. For this, we consider, preliminarily, the set $\{P_0,P_1,P_2,P_3\}$ and, following the procedure that we have indicated previously, we construct the curve $P(t)=(1-t)^2P_0+2t(1-t)P_1+t^2P_2$. In the similar way, we find that $P(t)=(1-t)^3P_0+3t(1-t)^2P_1+3t^2(1-t)P_2+t^3P_3$. In the figure 4 we show the graphic-geometric behavior. The quadratic and the cubic curve serve as a good example for discussing the visual development and understanding of a Bézier curve. (see figure 5).

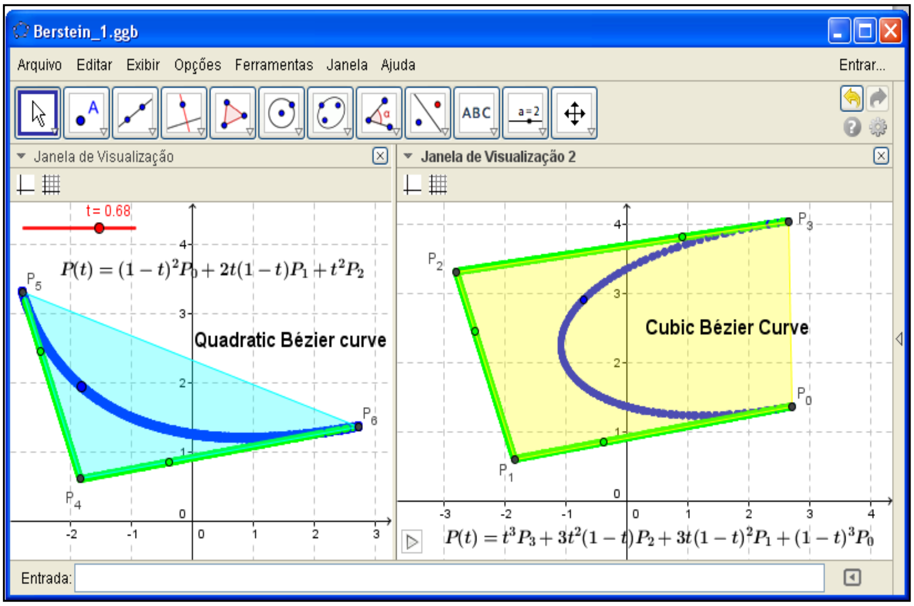

Figure 5. Visualization of a Quadratic Bézier and Cubic Bézier curve with DSG

First, is easy observe from (*), the following property: $B_{i,n}(t)=\binom{n}{i}t^i(1-t)^{n-i}=\binom{n}{i}t^i\sum_{k=0}^{n-k}(-1)^k\binom{n-i}{k}t^k=\sum_{k=0}^{n-k}(-1)^{k-i}\binom{n}{i}\binom{n-i}{k}t^{i+k}=$

$= \sum_{k=0}^{n-k} (-1)^{k-i} \binom{n}{i} \binom{n-i}{n-i} \cdot t^k = \sum_{i=k}^{n} (-1)^{k-i} \binom{n}{k} \binom{k}{i} \cdot t^k$. Similarly, we can show that each of these power basis elements can be written as a linear combination of Bernstein Polynomials. In fact, if we have a basis $\left[1, t, t^2, t^3\right]$ so, the Bernstein basis associated is described $\left[(1-t)^3, 3t(1-t)^2, 3t^2(1-t), t^3\right]$. Moreover, we can still verify the propriety previously indicated in (**), by the following algebraic procedure:

$$(1-t)B_{i,n-1}(t) + t \cdot B_{i-1,n-1}(t) = (1-t)\binom{n-1}{\text{k}} t^k (1-t)^{n-1-k} + t\binom{n-1}{k-1} t^{k-1}(1-t)^{n-1-(k-1)} =$$

$$= \binom{n-1}{\text{k}} t^k (1-t)^{n-k} + \binom{n-1}{k-1} t^k (1-t)^{n-k} = \left[\binom{n-1}{\text{k}} + \binom{n-1}{k-1}\right] \cdot t^k \cdot (1-t)^{n-k} =$$

$$= \binom{n}{k} \cdot t^k \cdot (1-t)^{n-k} = B_{n,k}(t)$$

From a geometric point of view, when we consider the figure 2, we can infer too that all member of the family of polynomials are linearly independent. We did not demonstrate this formal property, despite being true. On the other hand, regarding the properties earlier commented, we note that its description does not provide an easy description en geometric way, even with the aid of a mathematical software.

We consider the quadratic curve described in the follow manner $P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$. When we lead with four points in the plane, we reach the cubic of Bezier $Q(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$. Well, is easy to verify that $(1-t)^2 + 2t(1-t) + t^2 = 1 - 2t + t^2 + 2t - 2t^2 + t^2 = 1$. And with a similar analytical argumentation, we can get too $(1-t)^3 + 3t(1-t)^2 + 3t^2(1-t) + t^3 = 1 = \left((1-t)+t\right)^3 = 1$. This last equality is obtained by $1 = 1 \leftrightarrow (1-t) + t = 1 \leftrightarrow ((1-t)+t)^n = 1^n = 1,\ n \in \{2, 3\}$.

In the context of research in engineering, we can think a curve in terms of its center of mass. For example, we take five control points P_0, P_1, P_2, P_3 like in the figure below. We still admit that each mass varies as a function of some parameter t in the following manner: $m_0 = (1-t)^3, m_1 = 3t(1-t)^2, m_2 = 3t^2(1-t), m_3 = t^3$ and, easily we compute that $m_0 + m_1 + m_2 + m_3 = 1$. Well, we know the expression corresponding to the mass center of four points $P(t) = \dfrac{m_0 P_0(t) + m_1 P_1(t) + m_2 P_2(t) + m_3 P_3(t)}{m_0 + m_1 + m_2 + m_3} = \dfrac{m_0 P_0(t) + m_1 P_1(t) + m_2 P_2(t) + m_3 P_3(t)}{1}$.

Now, for each value of t , the masses assume different weights and their center of mass changes continuously. In the figure 6, we visualize for each value of t, that the curve is swept out by the center of mass. We can manipulate this dynamic construction and observe that, when $t=0 \therefore m_0 = 1$ e $m_1 = m_2 = m_3 = 0$. This argument indicates that this curve $P(t)$ passes through $P_0 = (a,b) \in IR^2$. Moreover, when we take $t=1 \therefore m_3 = 1$ e $m_0 = m_1 = m_2 = 0$ and this curve passes through $P_3 \in (g,h) \in IR^2$. Furthermore, we can verify that the curve is tangent to $P_0 - P_1$ and $P_3 - P_2$. In the figure 6, the variable masses $m_i(t)$ are called blending functions. While, the $P_i(t)$ are known as control points.

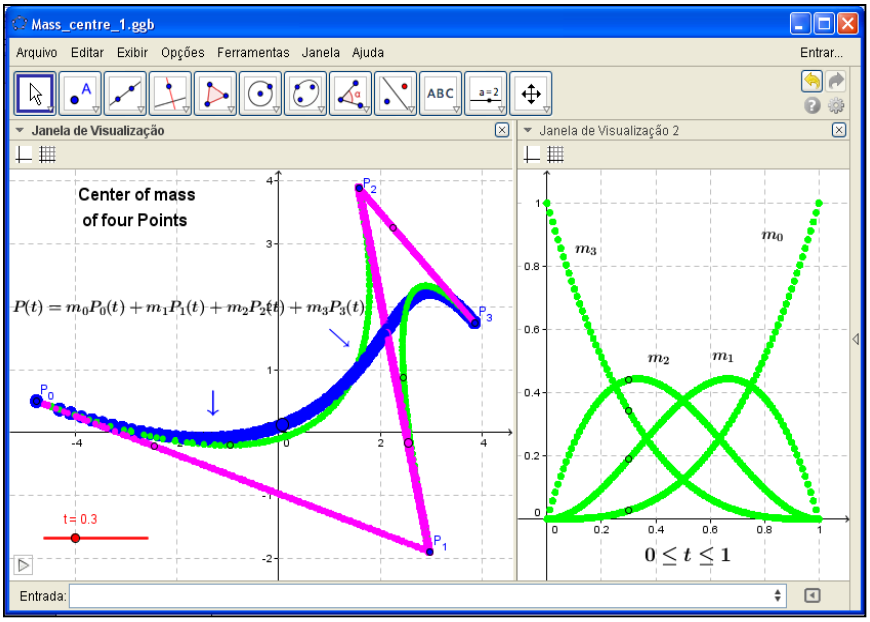

Figure 6. Visualization of the center of mass related to four points by DSG

From all these mathematical properties, we have a possibility to explore the actual technology. However, some routines are possible only when we explore a Computational Algebraic System – CAS (including in the Bezier curves and the Berstein´s polynomials), in a complementarily way (ALVES, 2014). In fact, if we take the set of control points: $P_0 = (1,0); P_1 = (4,2); P_2 = (2,4); P_3 = (0,2); P_4 = (-4,4)$. We just indicate in the figure 7 and we need to highlight the high operating cost when dealing with a large amount of these points (see figure 8).

In this dynamic construction, we have used the CAS Maple en virtue to determine, en according the Vainsencher´s algorithmic, a Bezier curve with five control points. We can visualize a blue curve below!

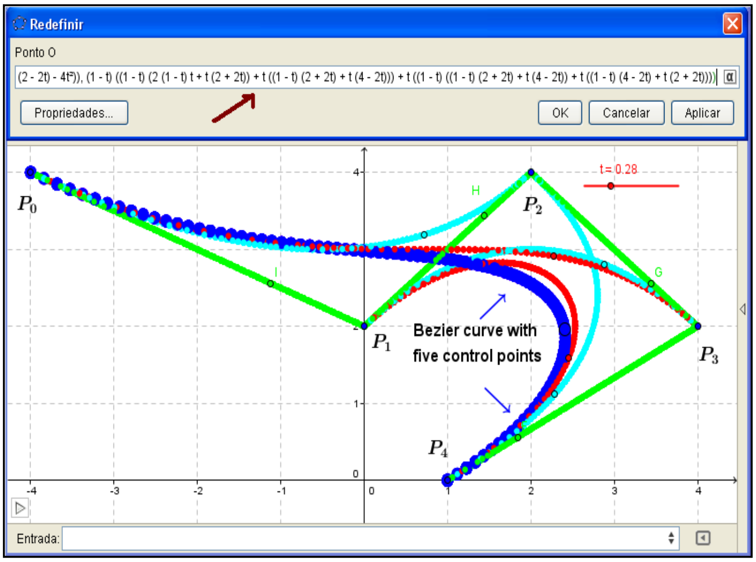

Figure 7. Some computations problems can be solved by DSG and the CAS maple

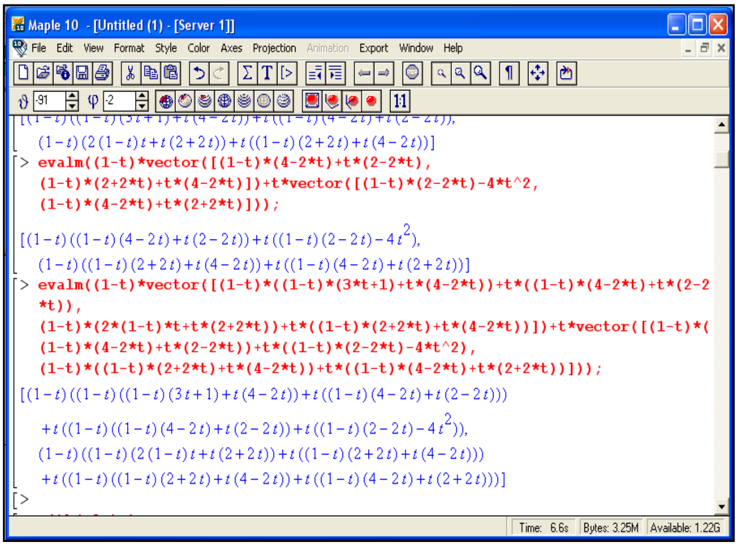

Figure 8. Some commands in CAS Maple describing the Bézier curve

We suggest to the reader the exploration of the CAS Maple. In the figure 8, we have take the same set of points

$P_0 = (1,0); P_1 = (4,2); P_2 = (2,4); P_3 = (0,2); P_4 = (-4,4)$, however, this computational description despite being in space IR^3, does not have the dynamic character like the earlier example (compare the figures 6 and 8).

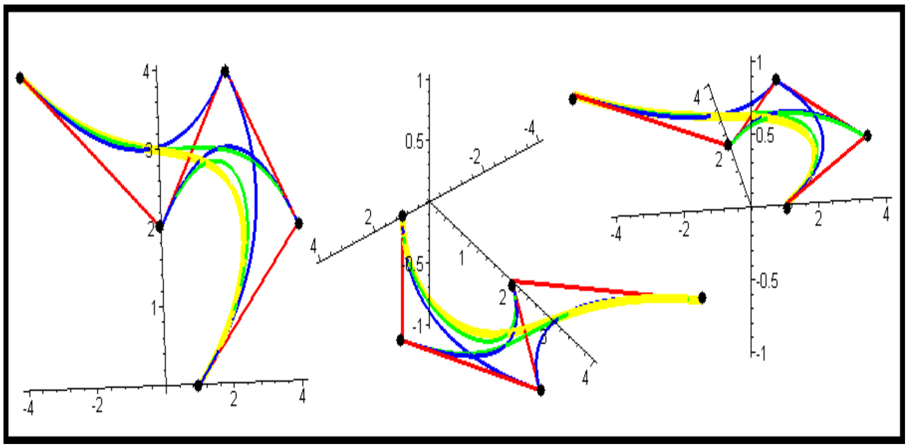

Figure 9. Visualization provided by CAS Maple in the tridimensional space

Final remarks

In several computational methods we identify the multiple uses of Bézier curves (like in Robotics and Computer Aided Geometric Design). In this work, we have emphasized some application en virtue to describe some basic graphical behavior of this concept and the Berstein´s polynomials too.

From historical point of view, the Bézier curve was formulated by Pierre Bézier, in 1962 and, approximately of this period, Paul De Casteljau developed the same curve. "The conic sections, the brachistochrone curve, cycloids, hypocycloids, epicycloids are all examples of very interesting curves that can be easily described and analyzed parametrically" (NEUERBURG, 2003, p. 1). On the other hand, the difficulties with these traditional examples of parameterization are the lack of applications associated with them in several Calculus Books (ALVES, 2014)

Moreover, several computational problems indicate a strong relationship between the Bernstein's polynomials and the Bézier curves. In fact, Bertot, Guillot & Mahboubi (2010) explain various situations. So, en virtue a didactic preoccupation, similarly the Vainsencher´s intention, we observe that some aspects that we have discussed here have a relevant pedagogical value, particularly with regard to intuitive and geometric understanding of a method briefly discussed in the context of Algebraic Geometry (VAINSENCHER, 2009, p. 117). Indeed, we can visually compare in the fig 9, an old and static description related to a Bezier curve.

Pouget (1995, p. 127) comments that "the Bezier and B-splines models are in ascendant complexity. However, this complexity is not necessary in several opportunities. In fact, the Bezier cubic has many applications". From this view, we can explore some visual properties (like figures 2, 5, 6), en virtue to promote a visual comprehension of it. Below, Pouget (1995) transmits some geometric and a visual meaning.

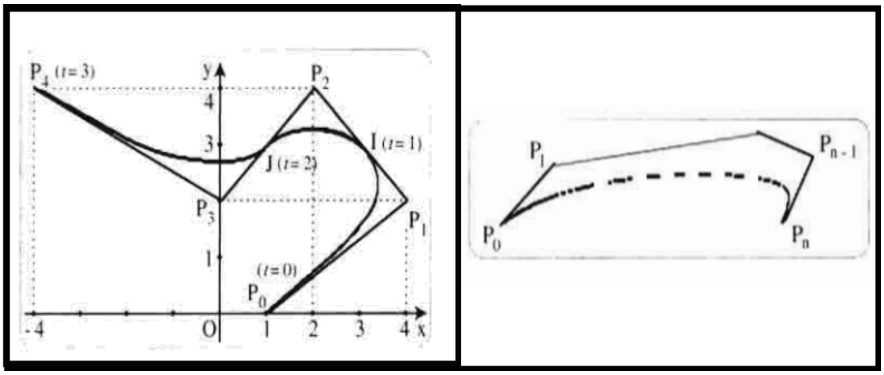

Figure 10. Pouget (1995, p. 123) describes an infinite approximation process

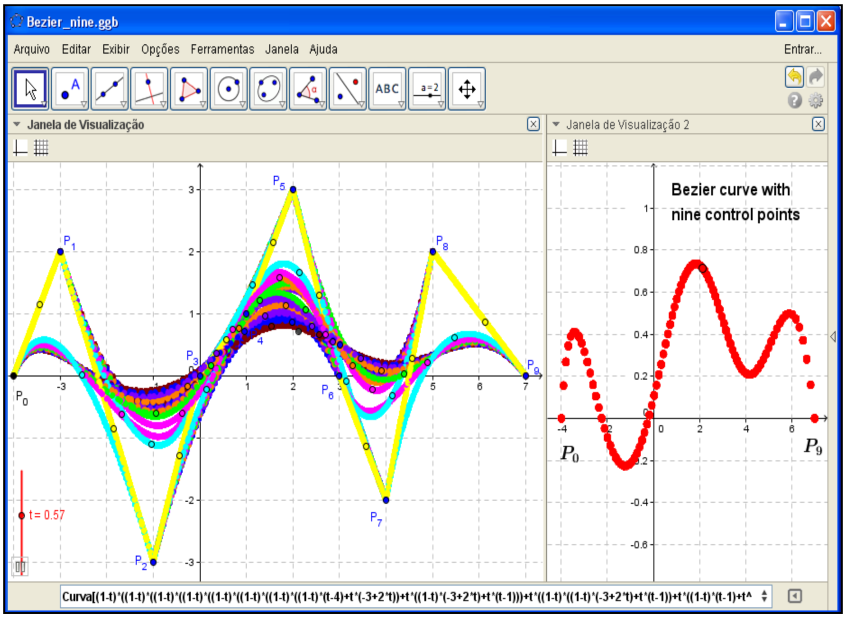

Figure 11. Visualizing a Bézier curve with nine control points provided by DSG and the CAS Maple

Finally, we show an arduous work en virtue to describe and visualize a Bézier curve with nine control points.

On the left side, we have used the Casteljau´s algorithmic formally described for $\begin{cases} P_i^r(t) = (1-t)P_i^{r-1}(t) + tP_{i+1}^{r-1}(t) \\ P_i^0(t) = P_i \end{cases} \left(1 \le r \le n \text{ e } 0 \le \text{i} \le n-r\right)$ en virtue to obtain the recursive curves. And on the right side, we obtain the final trajectory related a (red) curve from the set of control points $\{P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9\}$.

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