Construction of parameterized curves with the Dynamic System GeoGebra - DSG

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ABSTRACT: Parameterized curves constitute a compulsory content of study in academic locus at Brazilian universities. In this paper, we discuss a specific construction of this mathematical object supported by Dynamic System Geogebra – DSG. With this purpose, we describe several elements related to the construction of tree parameterized curves. At the end, we indicate some didactical moment en virtue its exploration at the academic level.

KEYWORDS: Parameterized curves, Construction, DSG - Geogebra, Visualization.

1 Introduction

In this paper we discuss some examples related to the construction of parameterized curves indicated by $\alpha(t),\beta(t),\delta(t)$, described for $\alpha(t)=(x(t),y(t))\in IR\times IR$, where $t\in IR$. We will show how the DSG Geogebra permits the exploration of mathematical qualitative properties extracted of the velocity and accelerator vectors. In fact, we will describe, in the next section, same didactical moments which can be developed in the academic Multivariable Calculus class.

In fact, we know that $\alpha'(t)=(x'(t),y'(t))\in IR^2$ represents the velocity vector related to the curve $\alpha(t)$. Moreover, we can obtain too the accelerator vector en virtue that $\alpha''(t)=(x''(t),y''(t))\in IR^2$. In fact, the behavior of the following symbol $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ will can provide a significant tool for to analyze the behavior of a tangent lines associated to a curve $\alpha(t)$. We still can study the formal quotient

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dy}{dt}}$ which can predict the existence of the inflection points (where we expect the changes relatively the concavity's direction (up $\bigcup$ and down $\bigcap$).

2 Some construction supported by DSG

Let us consider the parameterize curve $\alpha(t) = \left(\frac{2t}{1+t^4}, \frac{2t^3}{1+t^4} \right)$. Easily, we obtain that $\alpha(-t) = \left(x(-t), y(-t) \right) = \left(-x(t), -y(t) \right) = -\alpha(t)$, $\forall t \in IR$. We can get too $\alpha'(t) = \left(2\frac{(1-3t^4)}{(1+t^4)^2}, 2t^2 \frac{(3-t^4)}{(1+t^4)^2} \right)$. From this vector, we extract relevant information of this trajectory in the plane. In order to the condition $x(t) = 0$ and $y(t) = 0$, we infer that $1-3t^4 = 0 \leftrightarrow t = \pm 1 / \sqrt[4]{3}$, while the other value is $t = 0$ or $t = \pm\sqrt[4]{3}$. In this case, there is not a corresponding parameter to annul both components at the same time.

In the figure 1, we indicate the black vectors in direction of the axes. In the right side, we visualize the behavior of a tangent family to the curve $\alpha(t)$. We note that, when we consider the moving point A on this parametric curve and in the case that an approximation at the origin occurs, the software manifest limitations in describing their behavior.

From the vector $\alpha'(t) = \left(2\frac{(1-3t^4)}{(1+t^4)^2}, 2t^2 \frac{(3-t^4)}{(1+t^4)^2} \right)$, we compute $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2(3-t^4)}{(1-3t^4)}$ and analyze the behavior of a tangent family which we observe in the figure 1 (on the right side in green color). With the condition $t^2(3-t^4) = 0$ and $(1-3t^4) \neq 0$ we obtain the particular parameters which we have a tangent parallel to the axe Ox. Similarly, with the condition $t^2(3-t^4) \neq 0$ and $(1-3t^4) = 0$ we will have a tangent parallel to the axe Oy. At last, when we find parameters which satisfy $t^2(3-t^4) = 0$ and $(1-3t^4) = 0$ we identify the cusp points. Moreover, from the quotient $\frac{dy}{dx} = \frac{t^2(3-t^4)}{(1-3t^4)}$ we can conjecture the existence of an oblique asymptote.

The accelerator vector is described in the following manner $\alpha''(t)=\left(-\frac{24t^3}{(1+t^4)^2}-\frac{16(1-3t^4)t^3}{(1+t^4)^3},-\frac{24t^3}{(1+t^4)^2}-\frac{16(1-3t^4)t^3}{(1+t^4)^3}\right)=(x''(t),y''(t))$.

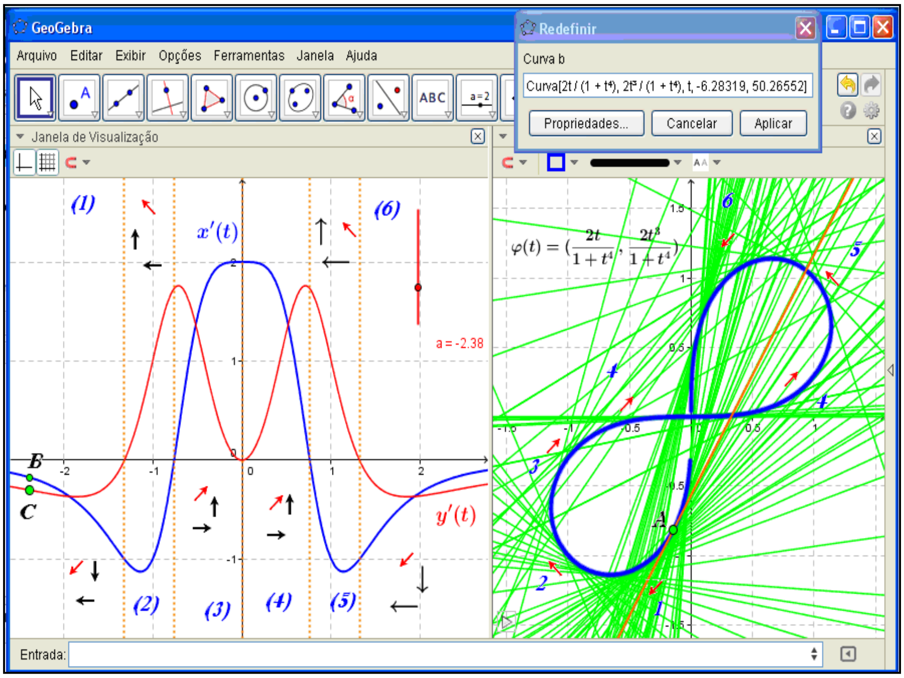

Figure 1. Visual analysis of the parameterized curve and the velocity components vector $\varphi'(t)=(x'(t),y'(t))$ and the software's limitations

In the figure 1, on the left side, we can easily see peaces the graph related to the component functions $x'(t)$ and $y'(t)$. In the figure 1-(1), we conclude that $x'(t)<0(\leftarrow)$ and $y'(t)<0(\downarrow)$. Therefore, the resultant vector (in red color) is indicated by ↙ In the figure 1-(2), we conclude that $x'(t)<0(\leftarrow)$ and $y'(t)>0(\uparrow)$. So, the resultant vector (in red color) is indicated by ↖. We analyze the peace of the graph in the figure 1-(3). We conclude in this case that $x'(t)>0(\rightarrow)$ and $y'(t)>0(\uparrow)$. So, the resultant vector (in red color) is indicated by↗. We follow the same verification to figure 1-(4). The resultant vector maintains the same behavior. In the figure 1-(5), we obtain that $x'(t)>0(\rightarrow)$ and $y'(t)<0(\downarrow)$. From these information, we indicate then ↖. Finally, we infer directly to the graph provided by DSG Geogebra that in the figure 1-(6), we find

$x'(t) < 0(\leftarrow)$ and $y'(t) < 0(\downarrow)$. And, from these facts, we infer the resultant vector $\swarrow$. At the end we compare all collected dates from the figure 1 with the analytical ones.

In the figure 1, in a first moment, is very difficult perceive the behavior of a concavity. However, with some basic commands of the DSG, we can identify the signal changes. Indeed, we take the following expression $\frac{d^2y}{dx^2} = \frac{d/dt\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{d/dt\left(\frac{(3t^2 - t^6)}{(1-3t^4)}\right)}{2\frac{(1-3t^4)}{(1+t^4)^2}} = \frac{(6t^9 + 12t^5 + 6t)(1+t^4)^2}{2(1-3t^4)^3}$. We observe that it's practically impossible analyze how the sign of this expression changes for the values t. On the other hand, we can formulate several conjectures from the figure 1, 2 and 3 based in the visualization.

In the figure 2, we visualize that in a neighborhood of the origin $(0,0)$ the velocity vector is null. However, we note the color changes correspondingly to the behavior of the tangents lines to the path below (on the right side). On the left side, we can compare the behavior of a tangent determinate by a moving point. With the DSG - Geogebra we can follow the points in the figure 3 for the same corresponding values for the parameter t.

The qualitative behavior which we can visualize in the figures 2 and 3 permits the understanding about the meaning of inflexion point (where we observe the change of direction of the concavity). Indeed, we can verify, by the direct manipulation the signal changes of the function $d^2y/dx^2(t)$. Moreover, we verify some limitations of the DSG. In fact, with intention to describe the curve below (figure 1, on the right side), we ca use Cartesian coordinates or a parametric form. In the first case, we could write $\left((3x-4y)^2 + (4x+3y)^2\right)^2 = 25\left((3x-4y)^2 - (4x+3y)^2\right)$. On the other hand, when we describe the curve in the parameterized form, we observe the (in color red) graphic related to the function $\frac{d^2y}{dx^2}(t)$ when the concavity passes from positive $(\cup)$ to negative $(\cap)$. Similarly, when the concavity passes from negative $(\cap)$ to positive $(\cup)$.

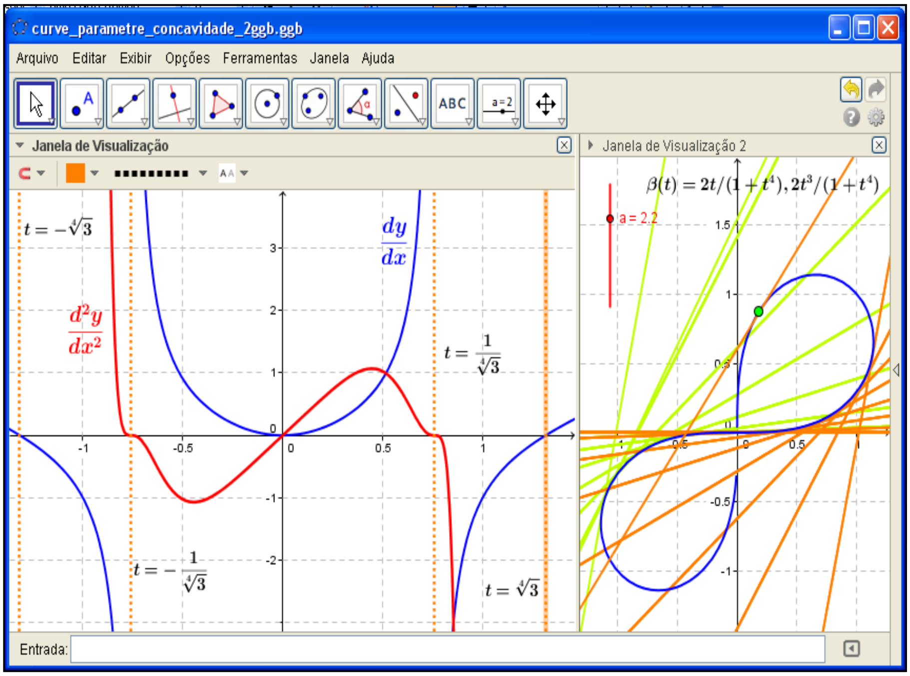

Figure 2. Study of the dy/dx e d^2y/dx^2 related to the concavity of the $\alpha(t)$

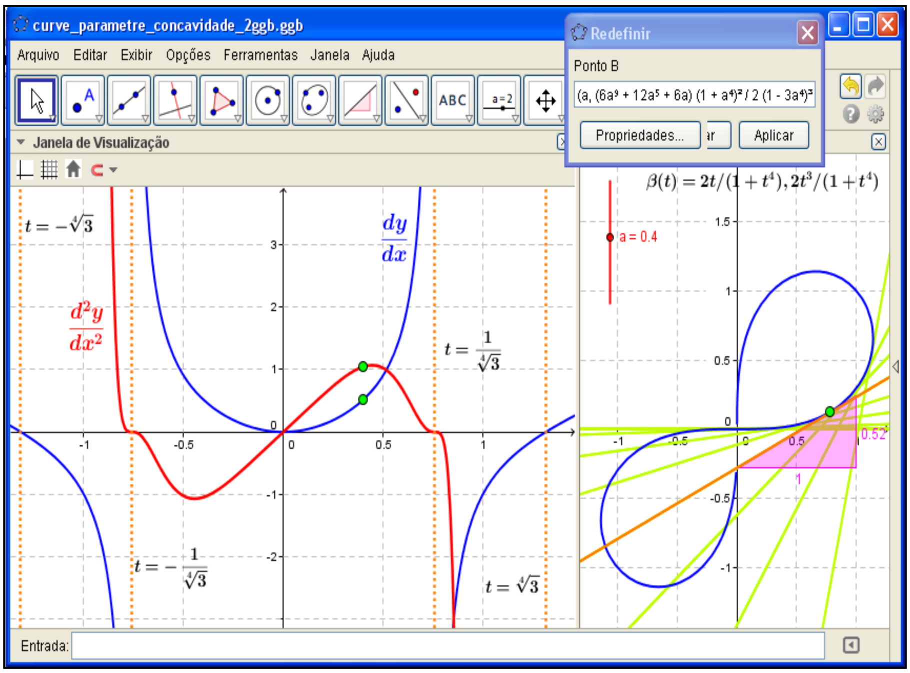

Figure 3. The behavior of concavity related to the signal of the function

Our next example is the parameterized curve $\beta(t)=\left(\frac{3at}{1+t^3},\frac{3at^2}{1+t^3}\right)$, com $a>0$. This expression is originating from the Cartesian equation $x^3+y^3=3axy$. We take the following relation $y=tx \therefore x^3+(t^3x^3)=3atx^2 \leftrightarrow x^3(1+t^3)=3atx^2 \leftrightarrow x(1+t^3)=3atx$. Finally, we express the variable $x(t)=\frac{3at}{1+t^3}$ and compute $y(t)=t\frac{3at}{1+t^3}$. Moreover, we observe $\lim_{t\to-1^+} x(t) \overset{a>0}{=} \lim_{t\to-1^+}\frac{3at}{1+t^3}\to-\infty$ while $\lim_{t\to-1^-} x(t) \overset{a>0}{=} \lim_{t\to-1^-}\frac{3at}{1+t^3}\to-\infty$. We can use the same arguments in the other cases indicated by the limits $\lim_{t\to-1^+} x(t) \overset{a<0}{=} \lim_{t\to-1^+}\frac{3at}{1+t^3}\to+\infty$ and $\lim_{t\to-1^-} x(t) \overset{a<0}{=} \lim_{t\to-1^-}\frac{3at}{1+t^3}\to-\infty$. In either case, we will determine the follow limit $\lim_{x\to+\infty}\frac{y(t)}{x(t)}=\lim_{t\to-1^+}\frac{y(t)}{x(t)}=-1=\lim_{t\to-1^-}\frac{y(t)}{x(t)}=\lim_{x\to-\infty}\frac{y(t)}{x(t)}$. Therefore, we can determine

$$b=\lim_{x\to+\infty} y(t)-(-1)x(t)=\lim_{t\to-1^-}\left[\frac{3at}{1+t^3}+\frac{3at^2}{1+t^3}\right]=3a\lim_{t\to-1^-}\left[\frac{t(1+t)}{1+t^3}\right]=$$

$=3a\cdot\lim_{t\to-1^-}\left[\frac{t(1+t)}{(t+1)(t^2-t+1)}\right]=3a\cdot\frac{-1}{3}=-a$. Similarly, we can get that $b=\lim_{x\to-\infty} y(t)-(-1)x(t)=-a \therefore y=-x+b=-x-a$, with $a>0$.

The velocity vector is described for $\beta'(t)=\left(\frac{3a-6at^3}{(1+t^3)^2},\frac{6at-3at^4}{(1+t^3)^2}\right)$. Now, we still obtain that $\frac{y'(t)}{x'(t)}=\frac{6at-3at^4}{3a-6at^3}=\frac{2t-t^4}{1-2t^3}$ which represents the slope vector related to a tangent line (see figure 4, on the right side).

In the figure 2, we indicate (on the right side) the graphical geometric behavior of this curve. We can conjecture its behavior when we have $1-2t^3=0\leftrightarrow t=\frac{1}{\sqrt[3]{2}}$ while $2t-t^4\neq 0$. On the other hand, when we take the condition $2t-t^4=t(2-t^3)=0$ and $1-2t^3\neq 0$ we obtain the real values $t=0$ and $t=\sqrt[3]{2}$. The values corresponding to the first, we determine the points of the trajectory where the tangent line will be parallel to the y-axis. For the same reason, we determine the points of the trajectory where the tangent line will be parallel to the x-axis.

Then, we consider the quotient $\dfrac{d^2y}{dx^2} = \dfrac{1}{3a}\dfrac{(14t^6 - 20t^3 + 2)(1+t^3)^2}{\left(1-2t^3\right)^3}$. In this case, we find a difficulty in trying to analyze the sign of this expression directly from their analytical expression. But, when we observe the figure 5, we can explore carefully each part of this trajectory. In fact, in a neighborhood of a point B, we related the behavior of the function $\dfrac{d^2y}{dx^2}(t)$ with this particular trajectory. Again, in the figure 4, we developed visual relationships between the behavior of the velocity vector $\beta'(t)$. We have listed each part of the trajectory by the numbers 1, 2, 3, 4 and 5 (fig. 4). This technique allows the identification of growth/degrowth of a movable point on the trajectory $\beta(t)$ that we have indicated on the right side in blue color.

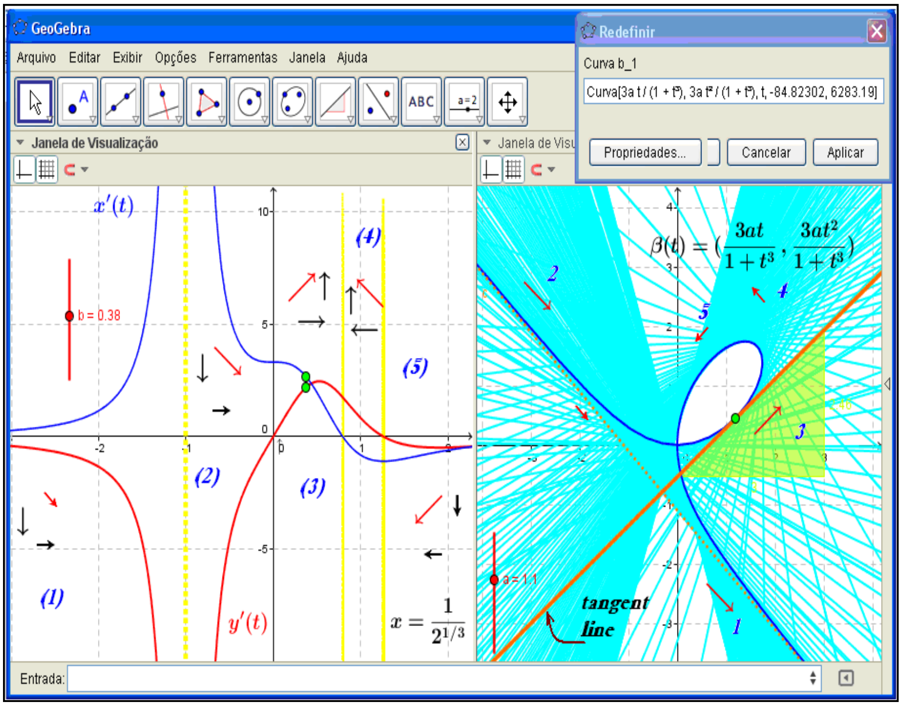

Figure 4. Visual analysis of the parameterized curve and the velocity components vector $\beta'(t) = (x'(t), y'(t))$

Similarly that we have showed in the figure 1, we will describe the same with the aim to understanding the behavior of the resulting velocity vector. In the figure 5, we indicate the graphics related to the functions $\dfrac{dy}{dx}(t)$ and $\dfrac{d^2y}{dx^2}(t)$. With some moving points (A, B and C), we can explore the signs changes of these functions.

So, similarly to the one variable Calculus, when we have $\frac{d^2y}{dx^2}(t) > 0$ we perceive the tangents family is over the parameterized curve (fig. 5, on the right side).

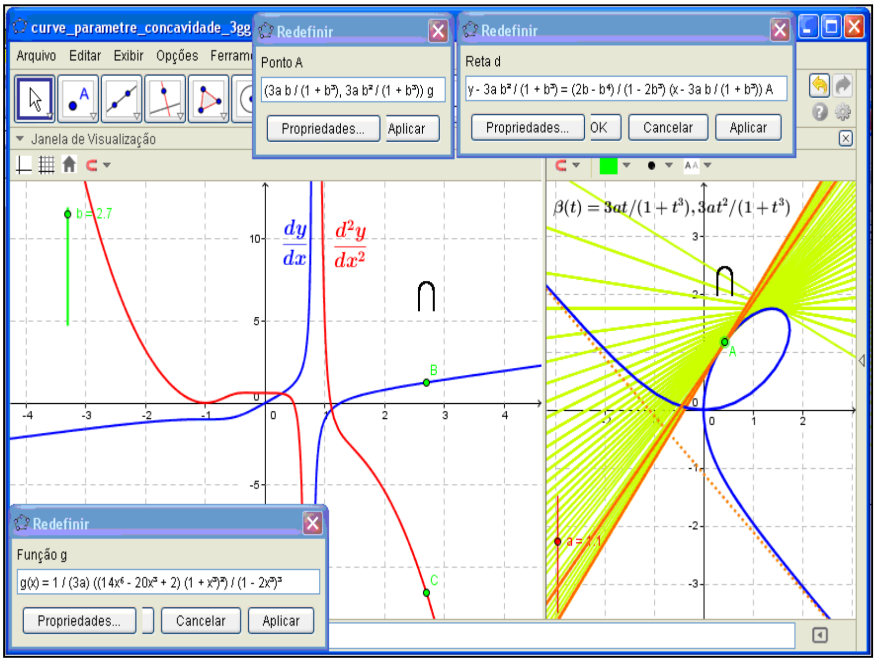

Figure 5. Description of the family of tangents and the concavity behavior

From the visual date in the figures 4 and 5, we can compare several mathematical properties which can be represented in a graphical-geometric manner. This description does not emphasize only the analytical procedures.

In the next example, we will consider $\delta(t) = \left(\int_0^t \frac{\cos(x)dx}{1+x^2}, \int_0^t \frac{\sin(x)dx}{1+x^2} \right)$ with $t > 0$. Again, we infer some basic properties like $x(-t) = -x(t)$ and $y(-t) = y(t)$. From this, we conclude that $\delta(t)$ is symmetric relatively the x-axes. On the other hand, we compute the velocity vector $\delta'(t) = (x'(t), y'(t)) = \left(\frac{\cos(t)}{1+t^2}, \frac{\sin(t)}{1+t^2} \right)$. Wit the DSG Geogebra, we can describe the graphical-behavior these elements. Below we show its trace by a moving point. However, not directly to this analytical expression $\delta(t)$.

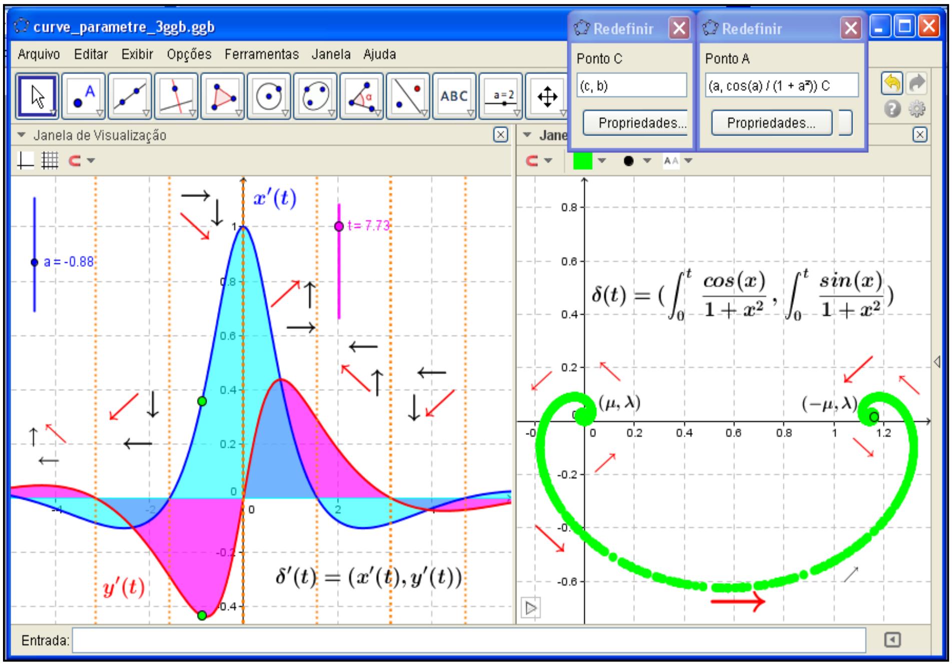

Figure 6. Visualization of the graphical-behavior of a parameterized curve $\delta(t)$

With intention to describe the analytical behavior for the $\delta(t) = \left(\int_0^t \frac{\cos(x)dx}{1+x^2}, \int_0^t \frac{\sin(x)dx}{1+x^2} \right)$, we will take:

$$\left| \int_0^t \frac{\cos(x)dx}{1+x^2} \right| \le \int_0^t \left| \frac{\cos(x)dx}{1+x^2} \right| = \int_0^t \frac{|\cos(x)|\, dx}{1+x^2} \le \int_0^t \frac{1dx}{1+x^2}.$$

Similarly, we write too $\left| \int_0^t \frac{\sin(x)dx}{1+x^2} \right| \le \int_0^t \frac{1dx}{1+x^2}$. Now, since we show in the figure 6, the convergence of the integral $\int_0^\infty \frac{1dx}{1+x^2} < \infty$, we obtain that $\mu = \int_0^\infty \frac{\cos(x)dx}{1+x^2} \in IR$ and $\lambda = \int_0^\infty \frac{\sin(x)dx}{1+x^2} \in IR$ are absolute convergent. From the properties that we have mentioned, we indicate the following points $(\mu, \lambda) \in IR^2$ and $(-\mu, \lambda) \in IR^2$. Easily, we observe the character of symmetry in the figure 6 (on the right side).

We can still compute $\delta''(t) = \left(-\frac{\sin(t)}{1+t^2} - \frac{2t\cos(t)}{(1+t^2)^2}, \frac{\cos(t)}{1+t^2} - \frac{2t\sin(t)}{(1+t^2)^2} \right)$ that indicate the acceleration vector. Moreover, we will define the following sequence

$\alpha_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin(x)}{1+x^2}dx$, with condition $\left(\alpha_n\right)_{n\geq 0}$. We restrict our analysis to the interval $[n\pi,(n+1)\pi]$ and we take:

$$\left|\alpha_n\right| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{1+x^2}dx \underset{dx=dy}{\overset{x=y+\pi}{\Rightarrow}} \left|\alpha_n\right| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin(y+\pi)|}{1+(y+\pi)^2}dy = \int_{n\pi}^{(n+1)\pi} \frac{|\sin(y)|}{1+(y+\pi)^2}dy$$

Now, we consider: $x \in (n\pi,(n+1)\pi) \therefore \frac{1}{1+(n+1)^2\pi^2} < \frac{1}{1+x^2} < \frac{1}{1+n^2\pi^2}$ (*). We take the integral sign in (*) and get the following inequalities:

$\frac{1}{1+(n+1)^2\pi^2}\int_{n\pi}^{(n+1)\pi}\left|\sin(x)\right|dx < \left|\alpha_n\right| < \frac{1}{1+n^2\pi^2}\int_{n\pi}^{(n+1)\pi}\left|\sin(x)\right|dx$. Immediately, we write $\int_{n\pi}^{(n+1)\pi}\left|\sin(x)\right|dx = \left|\int_{n\pi}^{(n+1)\pi}\sin(x)dx\right| = 2$.

These relation permit to infer that $\left|\alpha_{n+1}\right| < \left|\alpha_n\right|$ and $\frac{2}{1+(n+1)^2\pi^2} < \left|\alpha_n\right| < \frac{2}{1+n^2\pi^2}$ (**). Moreover, we still observe that $\alpha_n = (-1)^n\left|\alpha_n\right|$ and the final conclusion $\lim_{n\to+\infty}\alpha_n = 0$ (see (**)).

We note that in the interval $[n\pi,(n+1)\pi]$, we can obtain that $sen(x) = (-1)^n\left|sen(x)\right|$. We write that:

$$y((k+2)\pi) - y(k\pi) = \int_{k\pi}^{(k+2)\pi}\frac{\cos(x)dx}{1+x^2} = \alpha_{k+1} + \alpha_k = \left(-1\right)^k\left(\left|\alpha_k\right| - \left|\alpha_{k+1}\right|\right)$$

We should add some comments regarding of the curve indicated by $\delta(t) = \left(\int_0^t \frac{\cos(x)dx}{1+x^2}, \int_0^t \frac{\sin(x)dx}{1+x^2}\right)$. In fact, we can not obtain directly from this expression the graphical behavior for it. In the figure 6 (on the left side), we show the area´s contribution under the components functions. From this, we designate a moving point $\left(\underset{\substack{area´s\\ contributions}}{blue}\text{ color}, \underset{\substack{area´s\\ contributions}}{\text{pink}}\text{ color}\right) \in IR^2$ (see figure 6). We describe the trace of a moving point. In fact, the previous integrals are dependent on parameter and we find some cases that obtaining the primitive function is impossible (like the component $y(t) = \int_0^t \frac{\sin(x)dx}{1+x^2}$). In this case we use another technique of approach.

Final remarks

Undoubtedly, parameterized curves constitute a compulsory topic in the Multivariable Calculus. In this paper, we showed an approach that permits the student employs some knowledge related to the one variable Calculus. Indeed, we indicate the elements which permit the inference of the growth and the decrease of the velocity vector $\alpha'(t)$, from the date extracted of its components.

In the parameterized curves class, we have tree possibilities: parameterized periodic curves; parameterized rational curves and diverse curves (like translation invariants or homothetic, spirals). In which of these class, we have to modify the construction's graph steps.

Out methods indicates some particular didactical moments for to explore the parameterized graph´s construction with DSG Geogebra. In fact, we highlight the following steps: (i) from the parameterized curve $\alpha(t)$ we obtain the velocity and accelerator vectors; (ii) in according to the class of the parameterized curve, we must use the DSG for verify the graphics behavior of ${dy}/{dx}(t)$; (iii) in according to the class of the parameterized curve, we must use the DSG for verify the graphics behavior of ${d^2y}/{dx^2}(t)$. In the final step, we confront all dates. The analytical ones with the graphical ones.

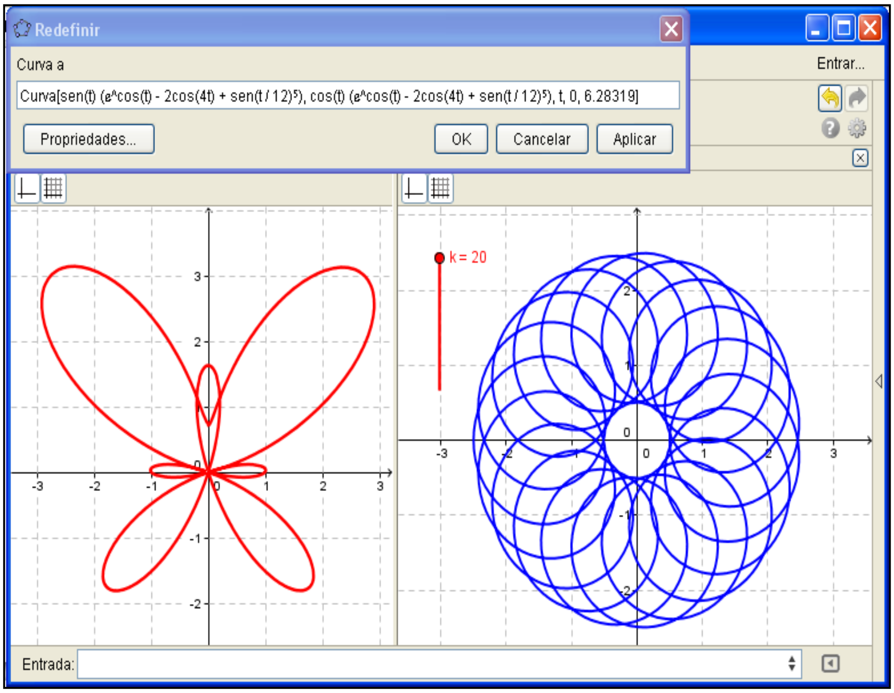

Figure 7. Some examples of parameterized curves with DSG Geogebra

Finally, we must be attentive to the style approach that prioritizes algorithmization in the Multivariable Calculus. In our approach, we emphasize the knowledge about the one variable Calculus. Indeed, in one variable Calculus, the students learnt that when we have $f'(x) > 0$ then the function $f(x)$ is increasing. Likewise, when we have $f'(x) < 0$ then the function $f(x)$ is decreasing (these properties are used in the figures 1 and 4). From these arguments, the students can observe that the prior knowledge remains valid, although we changed the context of working with the Calculus.

To conclude, we present below some examples of complex parameterized curves in the plane that can improve the cognitive perceptual skills of the students. Indeed our approach is limited in these cases shown. In this interesting topic, we find some examples of the works about the visualization provided by this DSG Geogebra (ALVES, 2014a, 2014b, 2014c, SURYNKOVA, 2012). Other examples can be finding in a historical context (YATES, 1947; ZDOROV, 1980).

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