Computational Technique for Teaching Mathematics – CT^2M: the quadratic forms´ case

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ABSTRACT: In this paper we indicate some elements en virtue to characterize a structured approach for teaching Mathematics in academic locus. We will refer as a Computational Technique for Teaching Mathematics - CT^2M . So, related to a specific content, we discuss some properties about quadratic forms. Indeed, we indicate the qualitative properties provided by the visualization of these conceptual objects in two and thee dimensional space. For this, we explore the potentialities of the DSG Geogebra and the CAS Maple. Our goal is to indicate an approach to Multivariable Calculus supported by the actual technology.

KEYWORDS: Computational Technique, Teaching, Geogebra, Quadratic forms.

1 Introduction

Undoubtedly, the branch of the study in Linear Algebra and Multivariable Calculus provides several barriers related to its teaching at the academic locus. The students have to study some complex and abstract topics in certain academic disciplines. In this paper, we will focus attention in the quadratic forms. This mathematical object makes possible some applications, for example, in Analysis and Calculus. However, en virtue an abstract character, how stimulate a visual understanding about it? With this scope, we will show the potentialities and differences related to the Dynamic System Geogebra – DSG in the teaching context.

Traditionally, we consider a application $\wp : IR^n \to IR$ described by the following expression $\wp(x_1, x_2, x_3, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \alpha_{ij} x_j$, where α_{ij} (*). On the other hand, we could take some particular cases and write: $\wp_1(x, y) = x^2 - y^2$,

$\wp_2(x,y) = xy$, $\wp_3(x,y) = x^2 + y^2$, $\wp_4(x,y) = 2x^2 - xy + \pi y^2$,
$Q_1(x,y,z) = \sqrt{3}z^2 + xy - xz + 2yz$, $Q_2(x,y,z) = x^2 + y^2 + z^2$, $Q_3(x,y,z) = 2x^2 - xy + \pi y^2$.
In these examples, we can discriminate the quadratic forms that allow us an interpretation in the plane and another is in space. Indeed, with the DSG we indicate the fig. 1 (in IR^2). On the other hand, with the Computational Algebraic System Maple, we indicate $\wp_1(x,y) = x^2 - y^2$ in the space. (fig. 2)

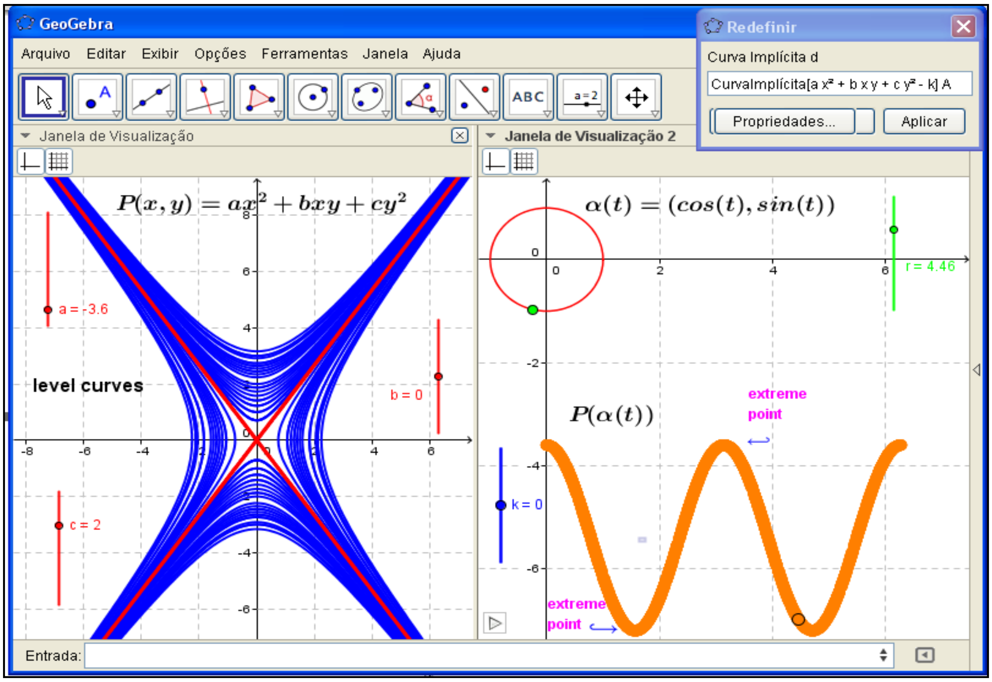

Figure 1. Visual and graphic interpretation of a quadratic form provided by DSG

In the figure 1, we visualize with the Geogebra´s help (on the left side), the behavior of a level set associated with $\wp_1(x,y) = ax^2 + bxy + cy^2$. On the right side, we can analyze its signal, directly to the orange graph. We have a tacit understanding about its behavior and the extreme point's localization. Indeed, we indicate a minimum an a maximum value assumed by $\wp_1(x,y)$. Our analysis needs all visual and numerical information. We observe, however, that the analysis in the figure 1 is restricting to bi-dimensional space. On the other hand, we show the figures 2 and 3 in IR^3.

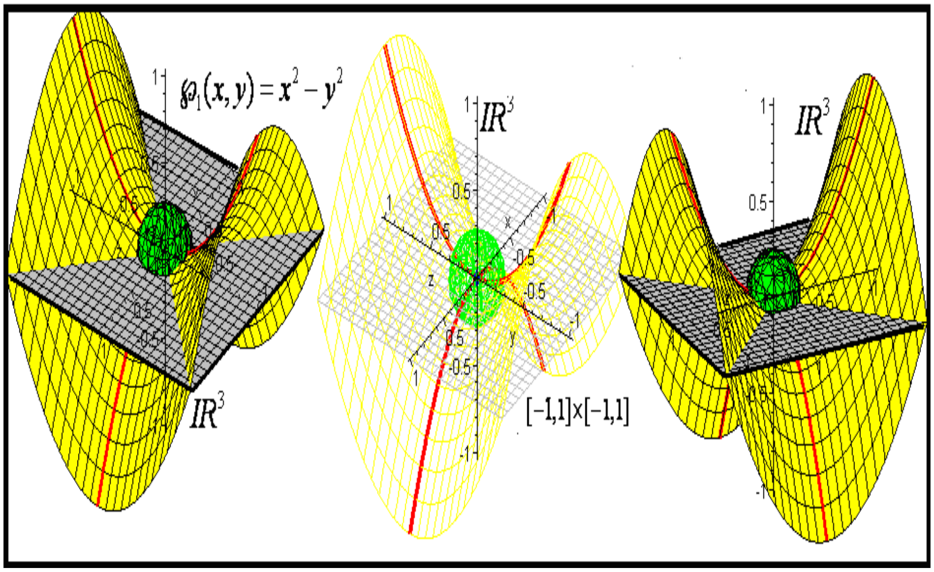

Figure 2. Visual and graphic interpretation of a quadratic form in the space IR^3

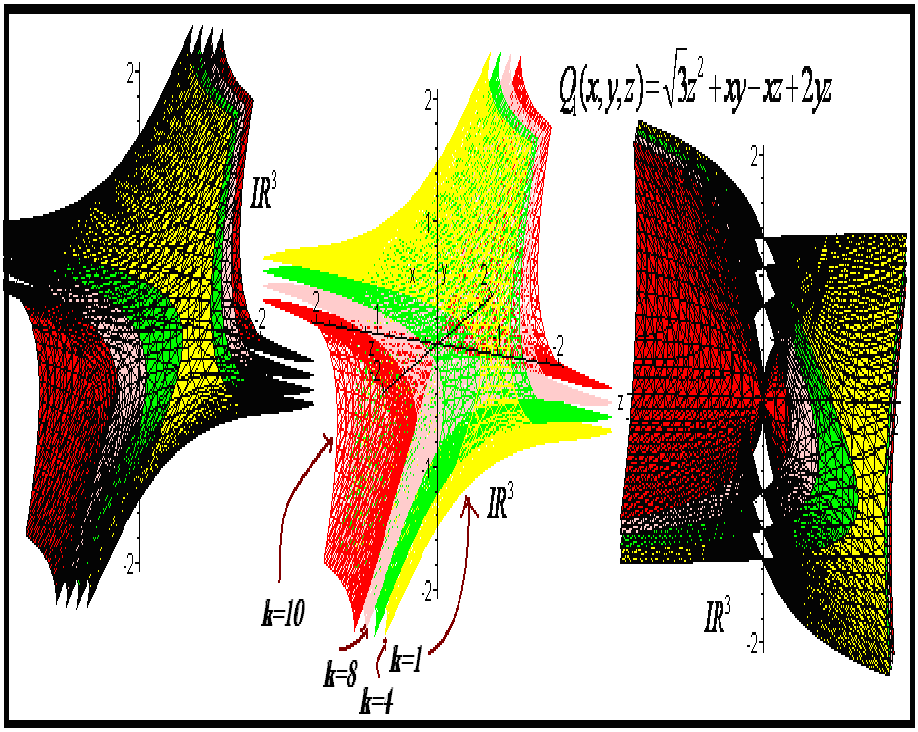

Figure 3. Level surfaces associated to a quadratic form in thee variables in IR^3

In the figure 2, we observe the graphic behavior of a quadratic form in the IR^3. While, in the figure 3, we indicate some level surfaces related to the form $Q_4(x,y,z) = 3x^3 - 4y^2 + \frac{3}{5}z^2 + 45xy - 87yz + 3xz$. In this particular case, the form $Q_4(x,y,z)$ can be associated a level surface in the IR^3. In the figure 3 we visualize some of them described by the condition $Q_4(x,y,z) = k$, where $k \in \{1,4,8,10\}$. From the figure 2, we could infer and conclude that $\wp_1(x,y) = x^2 - y^2$ is an undefined quadratic form (we see this formal definition in the next section). On the other hand, when we visualize the elements in the figure 3, how to identify and comprehend its behavior? How to classify the form indicated by $Q_4(x,y,z)$? From these and others questions, we will discuss preliminary some basic formal properties.

2 Some formal properties about quadratic forms

For all $\lambda \in IR$, the level set indicated by C_λ related to a quadratic form $\wp(x,y)$ is defined $C_\lambda = \{(x,y) \mid \wp(x,y) = \lambda\}$. From this set, we can declare that C_λ is a inverse image of the set $\wp^{-1}(\lambda)$. En virtue the $\wp(x,y)$ is a quadratic form, we can write $\wp(\lambda x, \lambda y) = \lambda^2 \wp(x,y)$. From this, we can obtain that: $(x,y) \in C_1 \leftrightarrow \lambda(x,y) \in C_{\lambda^2}$ and $(x,y) \in C_{-1} \leftrightarrow \lambda(x,y) \in C_{-\lambda^2}$.

In fact, if we take $(x,y) \in C_1$, we write $\wp(x,y) = 1 \therefore \lambda^2 \wp(x,y) = \wp(\lambda x, \lambda y) = \wp(\lambda(x,y)) = \lambda^2 \cdot 1$. From this, we compute that $\wp(\lambda(x,y)) = \lambda^2$. In the similar way, we consider $(x,y) \in C_{-1} \leftrightarrow \wp(x,y) = -1 \leftrightarrow \lambda^2 \cdot \wp(x,y) = -\lambda^2 \leftrightarrow \wp(\lambda(x,y)) = -\lambda^2$.

Finally, we will consider the following theorem.

Theorem 1: Any level set C_λ (with $\lambda \neq 0$) related to a quadratic form $\wp: IR^2 \to IR$ can be obtained from one of the two level sets C_1 or C_{-1} en virtue a uniform plane homothetic.

With the DSG Geogebra´s help, we can illustrate the last property and the geometric meaning related to a uniform plane homothetic transformation. In fact, in the figure 4, we can acquire a tacit understanding about the transformations when we vary the numbers $a,b,k \in IR$ in the quadratic form $\wp(x,y) = ax^2 + by^2 - k$. On the right side, for example, we indicate the variations corresponding to values $\wp(x,y) = cxy \;\; c > 0$ and c<0.

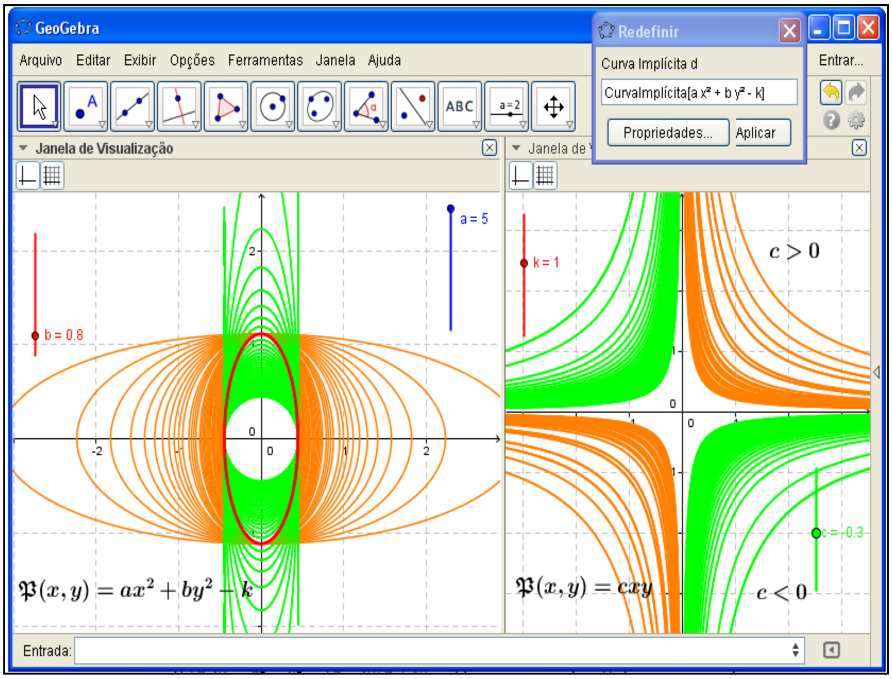

Figure 4. Visualization of the level sets related to a quadratic form from some variations of the parameters a, b, c and $k \in IR$

Formally, we know the standard classification related to quadratic forms. In fact, we know that a quadratic form is *positive definite* if $\wp(x_1, x_2, x_3, \ldots, x_n) > 0$ for all vector $(x_1, x_2, x_3, \ldots, x_n) \in IR^n$. In the same manner, we declare that a quadratic form is *negative definite* when $\wp(x_1, x_2, x_3, \ldots, x_n) < 0$ for all vector $(x_1, x_2, x_3, \ldots, x_n) \in IR^n$. At least, we have a *indefinite quadratic form* when $\wp(u) > 0$ and $\wp(v) < 0$, for some vectors $u, v \in IR^n$. In this paper, we will restrict our analysis to IR^2 and IR^3, en virtue the visualization's possibilities. In accordance with us, Mirsky (1955, p. 359) comments that "one of the main reasons for studying quadratic forms is their usefulness in the geometry of conics and quadrics; and the relation between quadratic operators and quadratic forms can, indeed, be aptly illustrated in this context". From this consideration, we will discuss some of this preoccupation and possibilities in the next section.

3 Computational Technique for Teaching Mathematics - CT^2M

Before discuss some aspects related to our mathematical goal, we should indicate some elements related to what we called in this paper by Computational Technique for Teaching Mathematics - CT^2M. Practically, we find some mathematical contents that permit a graphical and geometric interpretation. In our case, we are interest only in academic contents. On the other hand, in this locus, the mathematical complexity increases considerably. So, the first difficulty is to identify a content the permits a graphical and geometric interpretation in IR^2 and IR^3. Second aspect is recognizing that all software manifests some kind of limitation.

From this fact, we try to find some software that can not develop all mathematical interesting functions. At last, we describe a complementary structured situation according in a particular teaching context. The main goal is provide an understanding scenarios supported by 2D and 3D for the mathematical objects. So, we assume that this dimensional transition can promote and mobilize a tacit and intuitive knowledge for the student. In the next section, we bring some example related to CT^2M. We choose, in this work, the Computational Algebraic System CAS - Maple and the Dynamic System Geogebra – DSG.

4 Visualizing quadratic forms with the DSG

Why does one need quadratic forms? It's a good question. In fact, when we consider the teaching context, we find some crazy examples. Among the many uses of quadratic forms in Mathematics, we want to single out one of our attention in the Multivariable Calculus. In fact, from differential Calculus, we know when a point p is critical, we have $f'(p)=0$. On the other hand, from the Taylor's formulae that provides the approximation $f(p+h)\approx f(p)+\frac{1}{2}h^T\cdot D^2f(p)\cdot h$, where $p\in D\subset IR^n$. (**)

This kind of approximation indicates that we have an extreme point $x=p$ of the function f nearly closely with the signal of a two variables polynomial. Moreover, its degree is two. We still observe that $h^T\cdot D^2f(p)\cdot h$ is a term that represents a quadratic form (*). So, we acquire some advantages that to enable to analyze (locally) a function in terms of an associated quadratic form. With this aim, we can explore some examples.

We still note that the expression $h^T\cdot D^2f(p)\cdot h$ is similar to the following quadratic forms $Q_1(x,y)=x^2+5xy-7y^2=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 5/2\\ 5/2 & -7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ or $Q_2(x,y,z)=x^2-2xy+xz+7y^2-10yz+3z^2=\begin{pmatrix}x & z & y\end{pmatrix}\begin{pmatrix}1 & -1 & 1/2\\ -1 & 7 & -5\\ 1/2 & -5 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$. We observe that, in the first case, we can explore the DSG Geogebra, however, en virtue the existence of three variables, only the CAS Maple allow us to visualize some qualitative properties related to last form $Q_2(x,y,z)$. In the next proposition, we will indicate the relevance of the visualization en virtue to comprehend a teaching situation.

Proposition: Consider the quadratic form $\wp(x,y)=ax^2+bxy+cy^2$. If the bigger value related to this form, under restriction of a unit circumference centered at $(0,0)$ and radius $r=1$ is attained at the point $(x,y)=(0,1)$. Then, we have $b=0$.

In this case, we have that $\wp(-x,-y)=\wp(x,y)$, for all $(x,y)\in IR^2$. En virtue that symmetry, we will analyze only the plane region $y=f(x)=\sqrt{1-x^2}$. En virtue the hypothesis, we compute: $\frac{d}{dx}(\wp(x,y(x)))=\frac{d}{dx}(ax^2+bxy(x)+cy(x)^2)$

$=2ax+by(x)+bx\frac{dy}{dx}+2cy(x)\frac{dy}{dx}=2ax+by(x)-b\frac{x^2}{y(x)}-2cx$. Moreover, we write in the point $(x,y)=(0,1)\therefore dy/dx=-x/y\to dy/dx(0,1)=0$ and $d/dx(\wp(0,1))=0+b-b\cdot 0-2c\cdot 0=0\leftrightarrow b=0$. See the restriction $P(\alpha(t))$ in the figure 1, in the right side.

These formal arguments in no way assist the development of an intuitive (or a tacit) reasoning, although they are correct. With the DSG (see figure 1) we can extract a dynamic mathematical meaning related to this situation. On the other hand, we wish to point out elements that produce the visual meaning of extreme points in the IR^2 and IR^3. However, in the last case, we observe that a further difficulty related to the four dimensions.

For example, when we take the function $f(x,y,z)=x^2+y^2+7z^2-xy-3yz$ which posses its graph in the IR^4, it's impossible to visualize and extract some qualitative property. On the other hand, from the standard mathematical model, we know the hessian matrix $IH(x,y,z)=\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$ (***) and consider its behavior at the origin $(0,0,0)\in IR^3$. In the figure 5 we indicate the level surfaces $f(x,y,z)=k$, for $k\in\{1,3/2,2\}$. We observe some level surfaces in a neighborhood at the origin. Preliminarily, how to know that we have a maximum or a local minimum provided only the visualization? (see figure 5)

In fact, in the classical literature we find formal theorems that characterize the existence of the extreme points related to convex and concave functions class. However, based in the figure below, how to decide its nature near of the origin? We must observe, in this case, that we don't have none functions graph, only some level surfaces.

From analytical point, we know that $IH(0,0,0)=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 14 \end{pmatrix}$ and since

$D_1(0,0,0)= f_{xx}(0,0,0)=2>0$, $D_2(0,0,0)=\det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}=3>0$ and

$D_3(0,0,0)=\det(IH)=24>0$ the Hessian´s test tell us that exist a minimum local at the point $(0,0,0)$. In the figure 5, we observe a graphical-geometric model that not provide a intuitive model to make some relationship with which we are commonly adapted in terms of extreme points (in the IR^2 and IR^3).

In fact, we have several memories related to a scenarios related to a function, in one variable, that has a maximum local point (or a minimum). For this, we expect that the concavity of the graph of the function turned down (or turned up). In the similar way, we can recognize when we have a minimum local point.

In the case below, we find a model that contradicts what we imagine to be an extreme point. Indeed, in this case, we have not a function's graph from a global point of view. Other interesting example is the function:

$$f(x,y,z)=x^2+y^2+(x+y-3z)^2+z^4+2z^3-5z^2$$

which, by the analysis of its associated quadratic forms, in the critical points $(0,0,0),(-2,-2,-2),\left(1/2,1/2,1/2\right)$,

we can infer that $(-2,-2,-2)$ and $\left(1/2,1/2,1/2\right)$ are local minimum, while at the origin, we have a indefinite quadratic form.

Without the use of the technology, it´s impossible to develop a heuristic reasoning supported by the visualization.

We can visualize the level surfaces

$$x^2+y^2+(x+y-3z)^2+z^4+2z^3-5z^2=k,$$

with $k\in IR$ only with the CAS Maple.

For such class of functions, we can´t use the DSG.

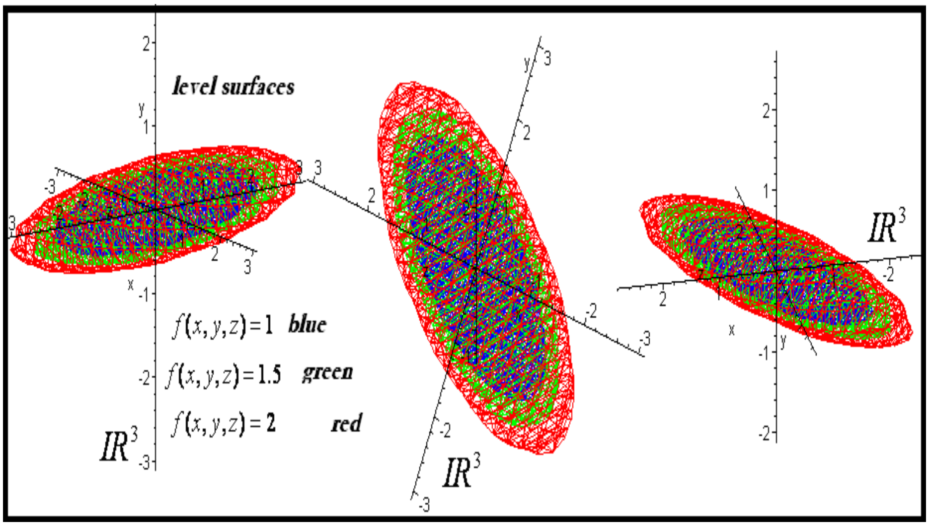

Figure 5. Visualization of level surfaces related to a quadratic form in thee variables

There are some cases like the function $f(x,y) = xy \cdot (1 - x^2 - y^2)$ that we verify multiple critical points. We don't analyze this kind of situation, but, the DSG allow us to verify the numerical behavior in each neighborhood that we can visualize in the figure 6. In this figure, we explore an arbitrary trajectory and follow the numerical behavior near each of these nine points that we indicate below. With this software, we can realize a numerical analysis related to each neighborhood, determine by the associated quadratic forms $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7$. (see figure 6)

Moreover, in the specialized literature, we find a strong relationship between the character of convexity and the hessian matrix numerical behavior (see expression (***) in the case $n = 3$). In fact, we illustrate the following theorem.

Theorem 1: Consider a function $f : U \subset IR^n \to IR$, $f \in C^2$ is defined in a convex sub-set $U \subset IR^n$. The, we have the equivalences: (i) f is a convex function in U if, only if, the hessian matrix $D^2 f(p)$ is definite positive for all $p \in U$; (ii) f is a convex function in U if, only if, the hessian matrix $D^2 f(p)$ is definite negative for all $p \in U$.

We will don't demonstrate this statement. For further details, the reader can see the classical literature (CALVO & DOYEN, 1982; EVES, 1066; JANICE, 1994; MIRSKY, 1955). However, we don´t have none qualitative visual element for to acquire an intuitive understanding in the IR^n (for $n \geq 4$).

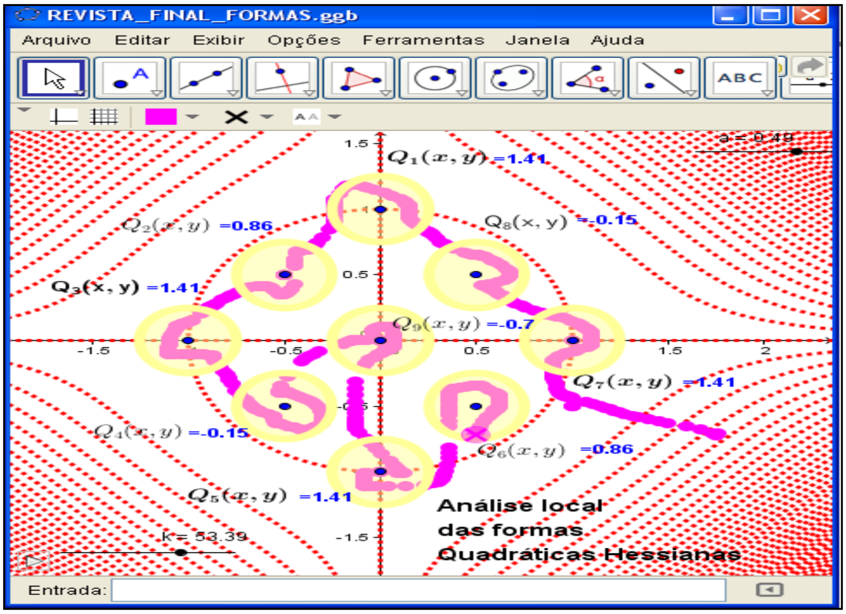

Figure 6. Numerical and topological identification of the neighborhood related to some quadratic forms supported by DSG

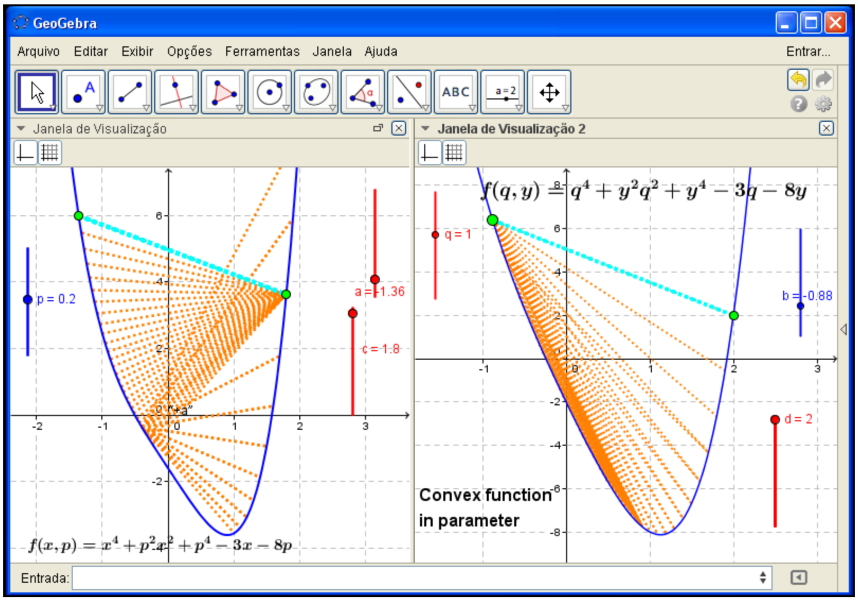

Figure 7. Visual description to a convex function and the its restrictions

On the other hand, in the figure 7, we indicate an approach to visualize a convex function in 2D. Indeed, we take the function $f(x,y) = x^4 + x^2y^2 + y^4 - 3x - 8y$. In the next step, we consider the two parameters restrictions $f(p,y)=p^4+p^2y^2+y^4-3p-8y$ and $f(x,q)=x^4+x^2q^2+q^4-3x-8q$. So, for each of its restriction, we verify the fundamental inequality related to the convexity described by $f((1-t)p+t\cdot q)\leq(1-t)f(p)+t\cdot f(q)$ or $f((1-t)p+t\cdot q)\geq(1-t)f(p)+t\cdot f(q)$. We can visually verify the first inequality in our particular case. (fig. 7)

We observe that in the theorem 1, that it's so difficult to acquire a geometric meaning directly from this mathematical statement. Contrary this situation, we show in the figure 7 and still conjecture something about the tangents lines behavior. In this case (on the left side, fig. 7), if we have a convex function, we can infer that the hyper plane is always below the graph. On the other hand, when we find a concave function, we infer that the hyper plane is always above the graph.

We have indicated some examples that we have the opportunity to explore a complementarily use of the DSG and the CAS Maple. In this way, we can find some particular manner to promote the visualization and the mathematical perception in the context $2D \overset{transition}{\rightarrow} 3D$. This kind the transition is essentially in the CT^2M. (ALVES, 2014).

Moreover, we indicate an application's context of the quadratic forms (in the study of the extreme point in Multivariable Calculus). In this situation, we observe that when the dimension $n\geq 4$, we can't visualize some properties, however, there are classical theorems (and analytical methods) that permit infer and make a final and exact conclusion, supported by some preliminary visualizations supported by the actual technology.

Final remarks

When we consider the Calculus teaching context, we find some crazy example that restricts all activities to an analytical frame. Moreover, without the actual technology, the visualization of the quadratic form behavior stays limited. On the other hand, based of a structured approach provided by the CT^2M, we can indicate for the student and the teacher, several elements supported by the visualization and a tacit mathematical comprehend.

Our approach indicate the relevance that the use of the DSG and the CAS Maple for the academic teaching. Through these use, we enable several possibilities of visualization of a quadratic form in two and tree variables. Moreover, through these date, we can establish some relationship with others classical theorems in Analysis or Linear Algebra (LANG, 1970; TAUVEL, 1970).

Indeed, in the past section, we have indicated some properties that are related to the quadratic form behavior. In the specific way, from the signal of the term in the expansion of Taylor (***), we can analyze the nature of an extreme point, however, in certain cases, we can´t visualize anything, en virtue that the condition $n\geq 4$. We have mentioned that, in the particular case $n=4$, we get only the level surfaces. (see fig. 3 and 4).

Finally, from an approach provided by the use of the two softwares, we described a Computational Technique for Teaching Mathematics - CT^2M. In fact, the CT^2M permits to explore the transition dimensional context $\left(2D \overset{transition}{\rightarrow} 3D \overset{transition}{\rightarrow} 2D\right)$. The visual phenomenon related to this transition can stimulate a tacit and heuristic understanding of the students.

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