Building counter – examples in teaching and learning of high school mathematics with the aid of Geogebra

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ABSTRACT: Geogebra is helpful dynamic mathematics software for teaching and learning of mathematics at school and university (see, for example, [An00], [Ve00]). In this article, we will discuss producing some high school mathematics counter - examples with the aid of Geogebra.
KEYWORDS: Geogebra, counter – examples, high school mathematics.

1. Introduction

In some cases, it is very difficult to build mathematical counter - examples in conventional teaching and learning of mathematics with pen - paper based environment, however, with the aid of dynamic mathematics software like Geogebra, we can easily produce mathematical counter – examples. In this article, we will give two examples in which we can see prominent role of Geogebra in making mathematical counter – examples.

For each example, firstly, we describe difficulties of creating mathematical counter – examples. Then we will discuss the way of producing mathematical counter - examples with the aid of Geogebra. Finally, we will present the mathematical solutions to the given problem.

2 Building high school mathematics counter - examples with the aid of Geogebra

In this part, we will discuss producing of high school mathematics counter - examples with the aid of Geogebra. We will present two examples in which it is difficult to create mathematical counter – examples in pen – paper based environment. Nevertheless, with the aid of Geogebra, we can easily build counter – examples to the given problems.

Problem 1. We are given a circle with the centre *O* and the radius *a (a>0).* Suppose that *AB* and *CD* are two diameters of the circle, and *AB* is perpendicular to *CD*. *M* and *N* are two points on the segment *DO* and *OC*, respectively such that *OM* = *CN*. The ray *AM* intersects the circle at *P*, different from *A*. Find *M* such that the angle *ANP* is a right angle. [Ng98a]

In the article [Ng98a], its author wrote that when he had tried to draw precisely and largely the figure of the problem, he believed that the angle *ANP* is always a right angle for an arbitrary point *M* on the segment *OD.*

Building counter – examples with the aid of Geogebra:

With the aid of Geogebra, it is easy to come up with the conclusion that there are only two positions of *M* (*M* coincides either *O* or *D*) such that the angle *ANP* is a right angle. The following are main commands of Geogebra we used to make counter – examples with the aid of Geogebra (see the Figure below):

- "*Circle with center through point*" *command* to draw an arbitrary circle with center *O* and through point *A*;
- "*Reflect about Point*" *command* (Reflects *A* through *O*) to locate point *B*;
- "*Segment*" command (Creates the segment *AB* between *A* and *B*);
- *"Perpendicular Line" command (*Creates the line through *O* perpendicular to *AB*).

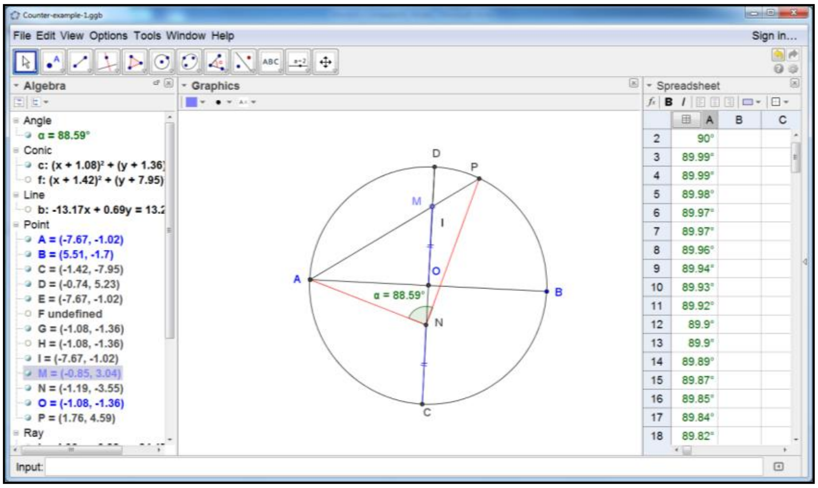
"Intersect" *command* (Yields the intersection points *C* and *D* of the circle with the line);

- "*Segment*" *command* (Creates the segment *CD* between *C* and *D*);
- "*Segment*" command (Creates the segment *OD* between *O* and *D*);
- *"Point on object" command* (Returns an arbitrary point *M* on the segment *OD*);
- "*Segment*" *command* (Creates the segment *CO*);
- *"Circle with Center and Radius" command* (Yields a circle with center *C* and radius *OM*);
- *"Intersect*" *command* (Yields the intersection points of the circle with center *C* and radius *OM* with the segment *CO*);
- *"Intersect" command* (Yields the intersection point *P* of the ray *AM* and the circle with center *O*);
- *"Angle" command* (measures the angle *ANP*).

If necessary, we use "*Rename*" *command*, "*Show Object*" *command*, *"Show Label" command* and so on.

When we drag *M* from *O* to *D*, we can see visually the measure of the angle *ANP*. From this visualization, we observe that in general, the angle *ANP is not a right angle* (except for two positions of *M*).

We can use "*Spreadsheet"* in Geogebra to put measure of the angle *ANP* in to the Spreadsheet.



Mathematical Solutions: We can use several ways to verify the prediction such that:

- Using coordinate method;
- Using vector method;
- Using indirect proof;

In this article, we briefly present a vector method solution to the problem.
We denote the measure of the segment OM by x ($0 \leq x \leq a$).

Then we have $\overrightarrow{AN} = \overrightarrow{AO} + \frac{a-x}{a} \cdot \overrightarrow{OC}$ (1)

Since the triangles AOM and APB are similar, we receive $AM.AP = AO.AB$.

Hence, $AP = \frac{2a^2}{\sqrt{a^2 - x^2}}$.

We see that $\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = \overrightarrow{AO} + \frac{x}{a} \cdot \overrightarrow{CO}$.

Therefore, $\overrightarrow{AP} = \frac{AP}{AM} \cdot \overrightarrow{AM} = \frac{2a^2}{(a^2 + x^2)} \left(\overrightarrow{AO} + \frac{x}{a} \cdot \overrightarrow{CO} \right)$ (2)

From (1) and (2), we have

$$\overrightarrow{PN} = \overrightarrow{PA} + \overrightarrow{AN} = -\frac{2a^2}{(a^2 + x^2)} \left(\overrightarrow{AO} + \frac{x}{a} \cdot \overrightarrow{CO} \right) + \overrightarrow{AO} + \frac{a-x}{a} \cdot \overrightarrow{OC}$$

$$= \left(1 - \frac{2a^2}{a^2 + x^2} \right) \cdot \overrightarrow{AO} + \left(a - x - \frac{2a^2 x}{a^2 + x^2} \right) \cdot \frac{1}{a} \cdot \overrightarrow{OC}$$

From the condition $\overrightarrow{AN} \cdot \overrightarrow{PN} = 0$, we obtain $(x-a) \cdot x \cdot \frac{(x^2 + xa + 4a^2)}{a \cdot (a^2 + x^2)} = 0$ (note that $\overrightarrow{AO}^2 = \overrightarrow{OC}^2 = a^2$ and $\overrightarrow{AO} \cdot \overrightarrow{OC} = 0$).

Since $\frac{(x^2+xa+4a^2)}{a\cdot(a^2+x^2)}$ is positive for all $a > 0$ and any x, we get $x = 0$ or $x = a$.

Consequently, there are two positions of *M* (*M* coincides either *O* or *D*) such that the angle *ANP* is a right angle.

Related problems:

Working with Geogebra suggests us the following (complicated) problem:

Assume that *a* as a constant. Let denote *x* the measure of the segment *OM* where $0 \le x \le a$. We consider the measure of the angle *ANP* as a function of variable *x*. Determine intervals of increase and decrease and identify extreme values of the function.

Problem 2. Let (*C*) be the graph of function $y = f(x) = x + \frac{1}{x}$. Find points *M*, *N* on (*C*) such that the *x* - coordinate of *M* is positive, the *x* - coordinate of *N* is negative, and the measure of segment *MN* is minimum.

The author of [Tu00] shows that high school students usually come up with mistaken prediction that when *M* and *N* are extreme of the function, the measure of the segment *MN* is minimum.

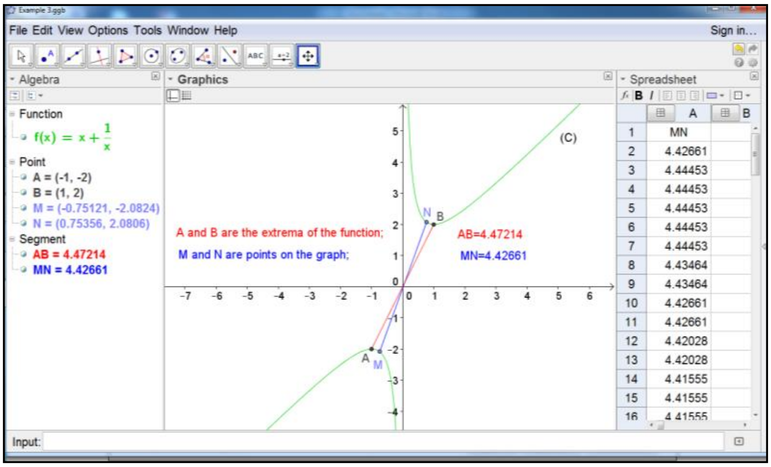
Conventionally, it is very difficult to verify that this prediction is true or false in pen – paper based environment.

Building counter - examples with the aid of Geogebra:

With the aid of Geogebra, we can easily produce mathematical counter – examples for the prediction. Here are the commands we use to create counter – example with the aid of the software:

- Using the "*Input Bar*" we directly enter $y = x + \frac{1}{x}$. The graph of the function is automatically displayed in the *Graphics View*.
- *"Extremum" command* to indentify the extrema *A* and *B* of the function;
- *"Point on Object" command* to locate points *M* and *N* on the graph;
- "*Distance or Length*" *command* to yield the distance between *A* and *B*;
- "*Distance or Length*" *command* to yield the distance between *M* and *N*;

When we drag the points *M* and *N*, we can recognize that there are the positions of *M*, *N* such that *MN* < AB (See the figure below).



Mathematical solution:

The solutions to the problem can be found in literature such as [Tu00].

Suppose that the x - coordinate of M and N are m and $-n$, respectively where m, n are positive real numbers.

We have $MN = \sqrt{[m-(-n)]^2 + \left[\left(m+\frac{1}{m}\right)-\left(-n+-\frac{1}{n}\right)\right]^2}$

$= \sqrt{(m+n)^2 + \left[(m+n)+\left(\frac{1}{m}+\frac{1}{n}\right)\right]^2} = \sqrt{(m+n)^2 + (m+n)^2\cdot\left(1+\frac{1}{mn}\right)^2}$.

Using the inequality of arithmetic and geometric means inequality, we get $(m+n)^2 \geq (2\sqrt{mn})^2$.

Therefore, $MN \geq \sqrt{4mn\cdot\left(2+\frac{2}{mn}+\frac{1}{m^2\cdot n^2}\right)} = \sqrt{8+4\cdot(2mn+\frac{1}{mn})}$.

By the inequality of arithmetic and geometric means inequality, we obtain $2mn+\frac{1}{mn} \geq 2\sqrt{2mn\cdot\frac{1}{mn}}$.

Hence, $MN \geq \sqrt{8+8\sqrt{2}}$.

As a consequence, the minumum of the measure of the segment MN is

$\sqrt{8+8\sqrt{2}}$ where $M\left(\sqrt[4]{\frac{1}{2}};\sqrt[4]{\frac{1}{2}}+\sqrt[4]{2}\right);N\left(-\sqrt[4]{\frac{1}{2}};-\sqrt[4]{\frac{1}{2}}-\sqrt[4]{2}\right)$.

3. Conclusion

With the above examples, we want to show that Geogebra is the power tool to making mathematical counter - examples in teaching and learning of high school mathematics. In our examples, however, Geogebra does not help us to find the mathematical solutions to the given problems.

References

[Ant00] **Antohe V.,** – *New methods of teaching and learning mathematics involved by Geogebra*, Geogebra International Journals of Romania, vol. 1, no. 2, 2010.

[Ng98a] **Nguyen M. H.,** – *Discussing the advantages coordinate of method in geometry*, Mathematics and Youth, vol. 2, 1998.

[Ng98b] **Nguyen V. B.,** – *An interesting mathematics problem - a counter - example*, Mathematics and Youth, vol.7, 1998.

[Tuo00] **Tuong M. M.,** – *Intuition in solving mathematical problems,* Mathematics and Youth, vol. 7, 2000.

[Vel00] **Velichová D.,**– *Geogebra tool for building conceptual understanding*, Geogebra International Journals of Romania, vol. 1, no. 1, 2010.