VISUALIZING THE BEHAVIOR OF INFINITE SERIES AND COMPLEX POWER SERIES WITH THE GEOGEBRA

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ABSTRACT: The study of the sequences, series of real and the complex numbers is compulsory in any academic book in Brazil. In this specific way, we discuss some properties related with these mathematical notions present in some books of Analysis. We give emphasis to several graphic-geometric properties with the use of dynamic system Geogebra. Thus we indicated the qualitative mathematical characters that become impractical in the teaching without the use of this software.

KEYWORDS: Visualization, Infinite series, Complex power series, Geogebra.

1 Some convergence criteria in *IR*

The study of the sequence of real numbers and series of real numbers is compulsory in all Real Analysis's book. In this branch, we find complex and formal definitions that can promote barriers for a heuristic and tacit understanding. In fact, in fig. 1 we visualize, on the left side, the behavior of partials sums of the *harmonic series* $\sum_{n\geq 1}^{\infty}\frac{1}{n}$ and the $\sum_{n\geq 1}^{\infty}\frac{(-1)^n}{n}$.

Specifically, we determine with the *software Geogebra*, the partial sums $\sum_{n\geq 1}^{50} 1/n$ and $\sum_{n\geq 1}^{50}(-1)^n/n$. In the first case, it's immediate to see that the values are increasing. Thus, when we observe the behavior of $\sum_{n\geq 1}^{50}(-1)^n/n$, we note that partials sums (in color green) decrease and oscillate around the axis of the abscissas. In this case, we may conjecture that this last series converge.

An necessary condition for the convergence is $a_n \to 0$, for $n \to +\infty$. We observe such condition in the first figure. And on the right side, we point the corresponding integral that allows us to conclude the behavior of the series.

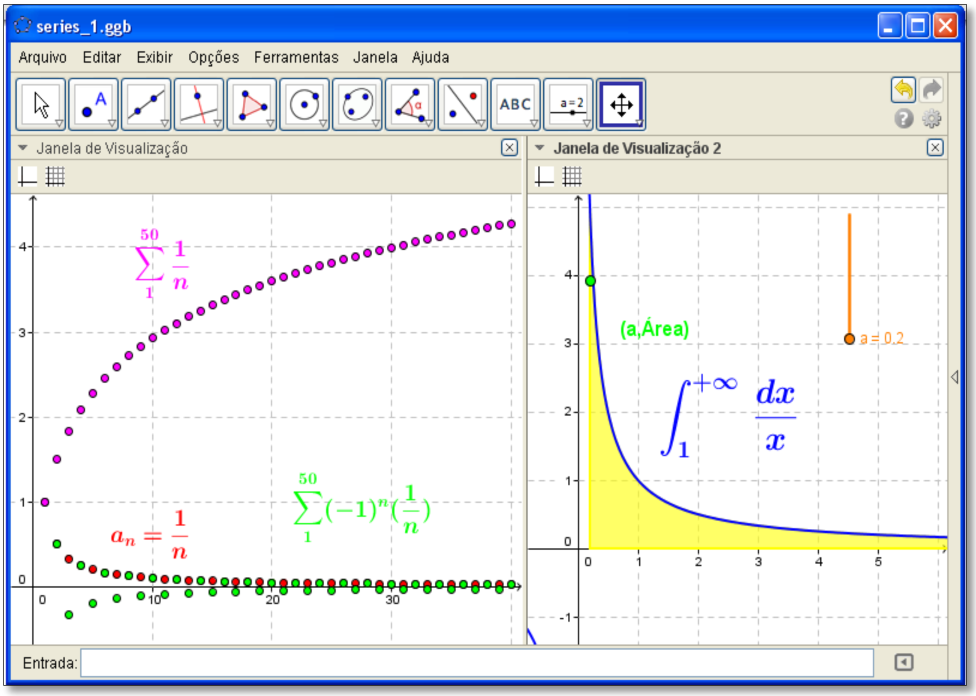

Figure 1. The relationship between the series and the improper integral

In the context of the Mathematical History, we find several examples related with the notions of the series (BOTTAZZINI, 2011). The authors like Hairer & Wanner (2008) explore the graphics for to signify the notion of convergence/divergence. We point out an example in the figure 2.

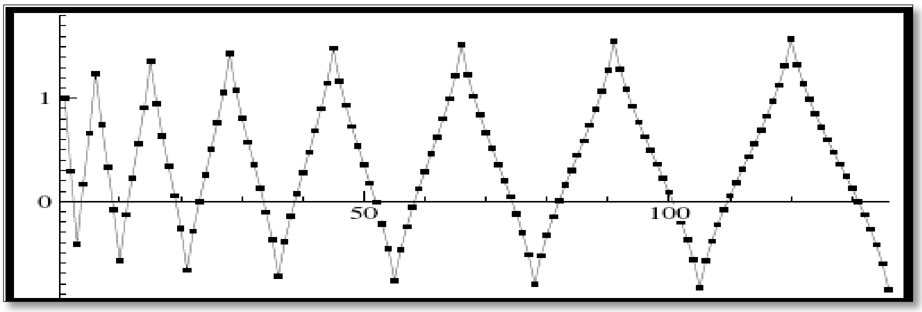

Figure 2. Hairer& Wanner (2008, p. 198) discuss an example of divergence studied by A. Cauchy (1759-1957) in 1821

Bartle & Sherbert (2000, p. 94) discuss the criterion of comparison for series. They point the $\sum_{1}^{\infty}\frac{1}{n^2}$, $\sum_{1}^{\infty}\frac{1}{n^2-n+1}$ and $\sum_{1}^{\infty}\frac{2}{n^2}$. They mention that the inequality $1/(n^2-n+1)\le 1/n^2$ is false for all $n\in IN$. In the figure 3, we observe that we can not use the *comparison test* involving the series $\sum_{1}^{\infty}\frac{1}{n^2}$ and $\sum_{1}^{\infty}\frac{1}{n^2-n+1}$. We can show that $\frac{b_n}{a_n}\to 1$ by means the behavior of the sequences in figure 3. On the other hand, with the aid the visualization, we can conclude that the partials sums $\sum_{1}^{50}\frac{1}{n^2-n+1}\le\sum_{1}^{50}\frac{2}{n^2}$ from the geometric-graphic frame. We still indicate some numerical values from the *spreadsheet* window of the *software Geogebra*. We can conjecture the numerical behavior of the partial sums of each series in figure 3.

We find several criteria of convergence for series of real numbers in specialized books. We can mark: the *Root Test*, the *Comparison test*, the *Leibniz Alternating series test* and the *Integral test*. We find the classical approach in some books for the *Integral test* that not emphasizes the geometric character (BARTLLE & SHERBERT, 2000; DAVIDSON & DONSIG, 2010; HAIRER & WANNER, 2008). Hairer & Wanner (2008, p. 259) attribute the *integral test* to Mclaurin in 1742. We illustrate this test by the figure 4. On the left side, we visualize that the behavior of the partial sums are increasing. On the other hand, when we explore only the geometric behavior of the integral, we could think that it converges (see yellow area).

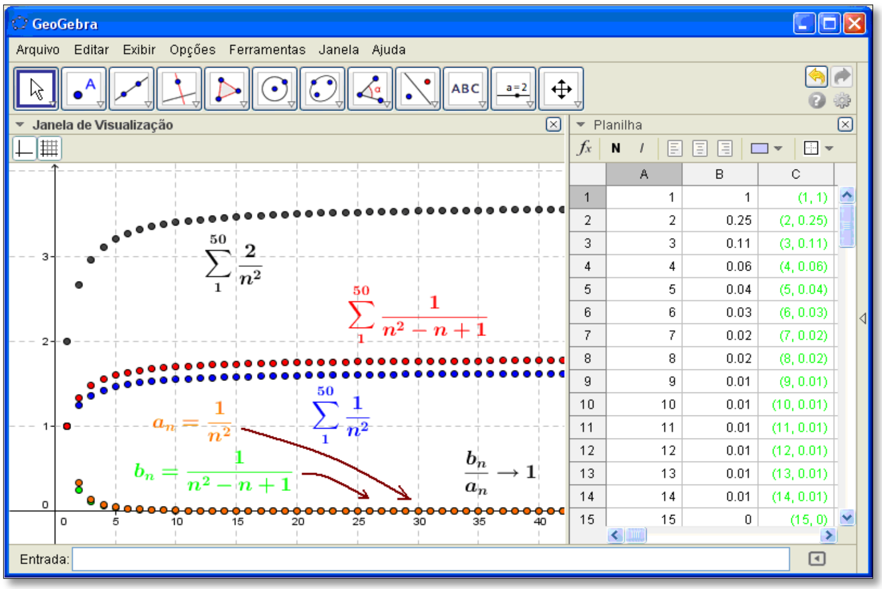

Figure 3. The software Geogebra allows to explore the numerical behavior of the series and the growth/degrowth of partial sums

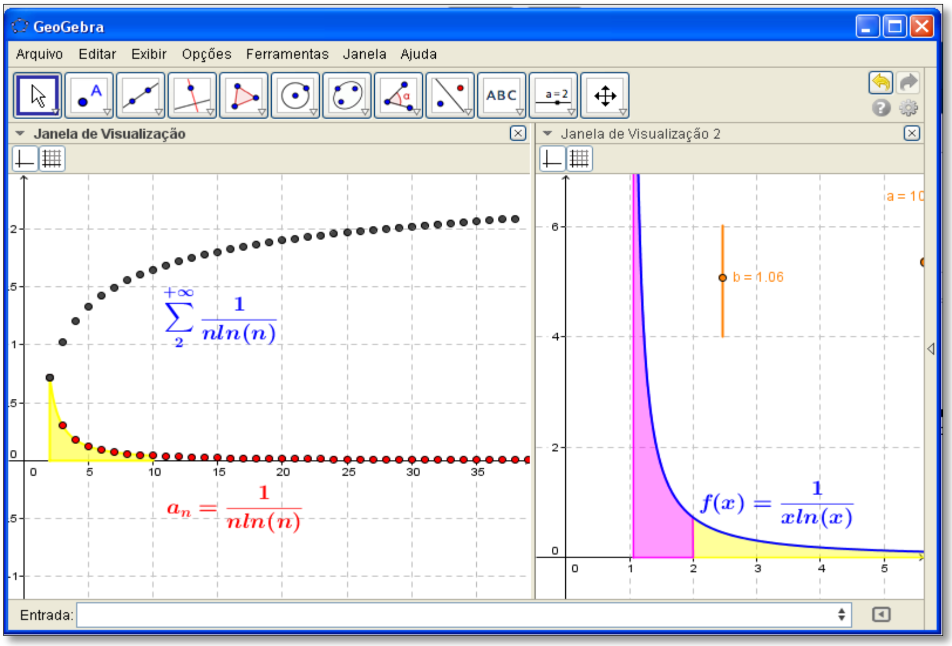

Figure 4. The graphic-geometric interpretation of the *Integral test*

In fact, when we look the area contributions (in yellow), we realize that they tend to decrease (in the right side, fig. 4). In this scene, the two regions below the function $1/(x \cdot \ln(x))$ (is positive, decreasing and continuous function in $(1,+\infty)$) exhibit different behaviors. To formulate a definite conjecture in despite the scene in the fig. 4, we must apply the *Integral test*. Soo, we can conclude that $\sum_{n\geq 1}^{\infty} 1/(n\ln(n))$ diverge, since the corresponding improper integral $\int_1^{\infty} 1/x \cdot \ln(x) = (\ln(\ln(x)))_1^{+\infty}$ also diverge.

Still in the figure 3, we can visualize some statements made by Bartle & Sherbert (2000, p. 94). In fact, we can not compare the terms of $\sum_1^{\infty} 1/n^2$, $\sum_1^{\infty} 1/n^2 - n + 1$. In the *spreadsheet window* (on the right side), we discriminate some numerical values. Our last example of series is the case of *p-series* $\sum_{n\geq 1}^{\infty} \frac{1}{n^p}$. We know that $\sum_{n\geq 1}^{\infty} \frac{1}{n^p}$ diverges, if $p < 1$ and converge if occurs that $p > 1$. By using of basic commands, including graphic animation, we can visualize the qualitative behavior of the contributions of area under the curve. In this heuristic way, we can understand this test.

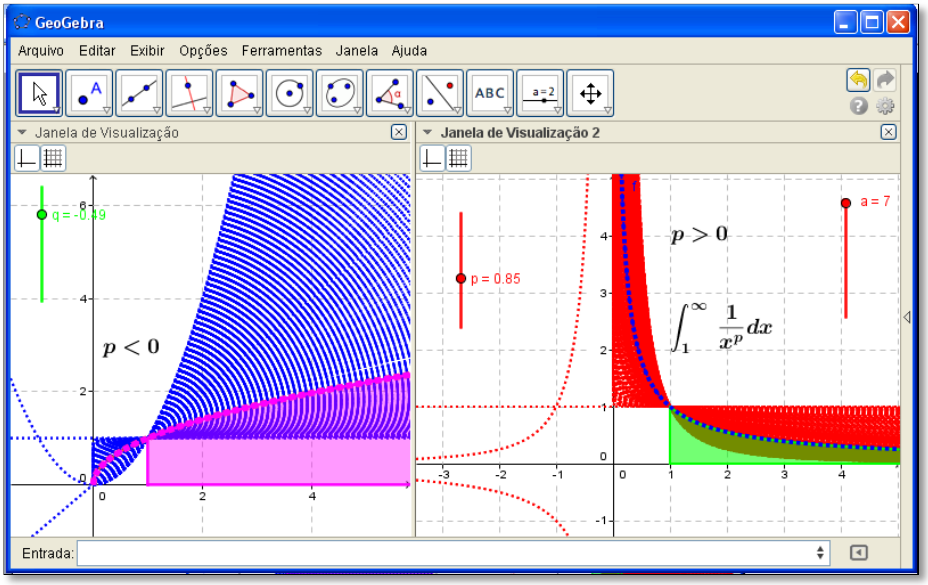

Figure 5. Family of functions related to the p-series

2 Some convergence criteria in *C*

Needham (2000, p. 67) explain that a complex power series $P(z)$ (centered at the origin) is an expression like $P(z) = c_0 + c_1 z^1 + c_2 z^2 + \cdots$ and their partial sums are just ordinary polynomials, described for $P_n(z) = c_0 + c_1 z^1 + c_2 z^2 + \cdots + c_n z^n$. The author preserves an approach that concedes a relevant role for the visualization. In the figure below, we must understanding that "once we reach a certain point $P_N(a)$ of the sequence $P_1(a), P_2(a), \ldots,$ all of the subsequent points lie within an arbitrary small disc of radius ε centered at A" (NEEDHAM, 2000, p. 67).

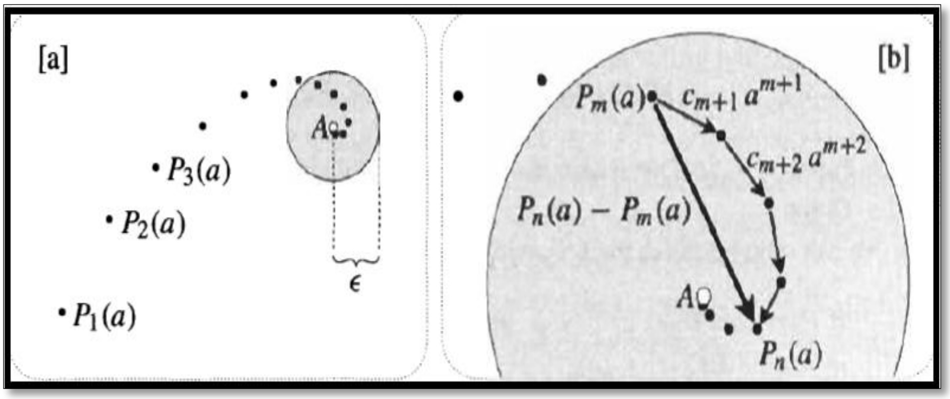

Figure 6. Needham (2000, p. 68) describe an heuristic interpretation for the convergence of complex power series

In the figure 6, we emphasize the approach of this author. We must observe that if $n > m > N$ then $P_m(a)$ and $P_n(a)$ both lies within the disc. We visualize the situation on the right side. This scene can promote the sense of the topological meaning related to the convergence. In the figure 7, we indicate the disc where we analyze the behavior of the partial sums described for $\sum_{n\geq 1}^{50} z^n$ and $\sum_{n\geq 1}^{70} z^n$.

We described the trace of the move point indicated by 'z'. It's possible identify a "ring of doubt", The geometric meaning of this term is marked by Needham (2000, p. 69).

Still in the figure 6, on the right side, we can visualize the property indicated by Needham (2000, p. 68) when the author states that "is $P(z)$ converges at $z = a$, then it will also converge everywhere inside the disc $|z| < |a|$.". In fact, we find in books of Complex Analysis that $\sum_{n\geq 0}^{\infty} z^n$ has radius of convergence equal to 1. Furthermore, converge for al $|z| < 1$

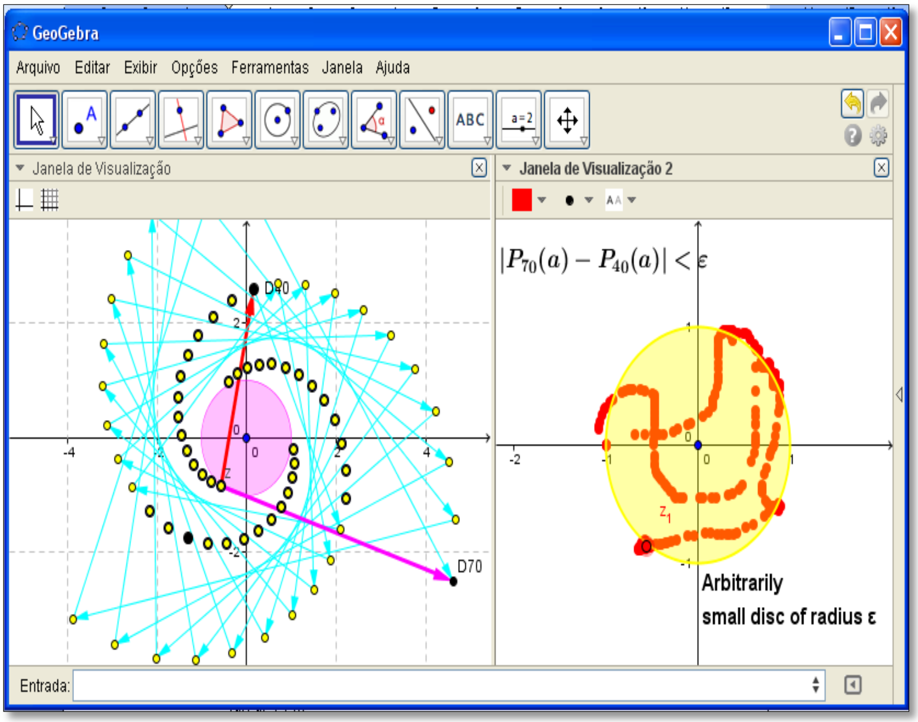

Figure 7. The software Geogebra provides the verification and the visualization of formal convergence properties in the Complex Analysis

In the figure 7, we show that at the same time that we drag de point (on the right side), the configuration and the behavior of the partial sums changes. In fact, we pointed a red vector and a pink vector (on the left side). We can visualize the graphic-geometric related property with the expression $\left|P_{70}(a) - P_{40}(a)\right|$ and we can predict when this difference becomes smaller or larger than the radius of convergence. In particular, in the figure 7, we see by considering some points $|z| > 1$ occurs that $\left|P_{70}(a) - P_{40}(a)\right| > 1$.

In figure 9, we consider the complex power series $\sum \frac{i^n (n!)^2}{(2n)!} z^n$ and, in this case, we can determinate the radius of convergence by the limit indicated for

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{n^2 + 2n + 1}{4n^2 + 5n + 2} = \frac{1}{4}.$$

In the figure 9, we consider the function $f(x) = \frac{x^2 + 2x + 1}{4x^2 + 5x + 2}$ and it´s restriction $f(n) = \frac{n^2 + 2n + 1}{4n^2 + 5n + 2}$, for $n \in IN$ determine a convergent limit.

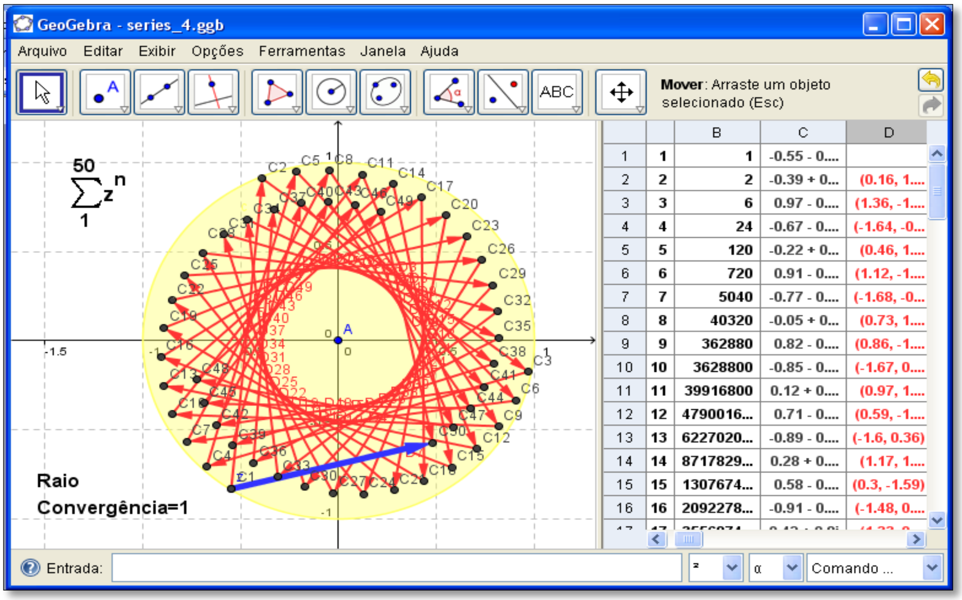

Figure 8. The software Geogebra provides the verification and the visualization of formal properties in the Complex Analysis

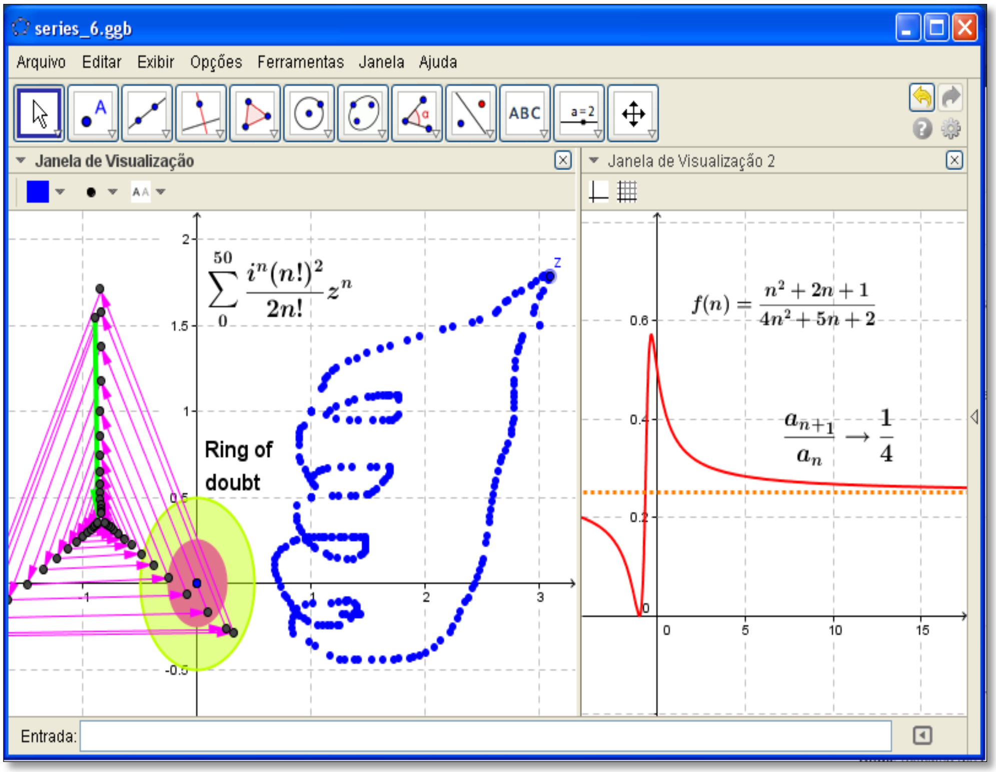

Figure 9. Exploring a region in the plane and dragging a point with the aim to comprehend the behavior of the complex power series

In the figure 8, on the left side, we indicate a disc with radius 1. When we drag the move point inside of this disc, we can observe that the partial sums $\sum_{n\geq 1}^{50} z^n$ corresponds a resultant blue vector, whose the length is smaller than $R = 1$.

Although, when we approach the point to the border of the disc, the sums manifests an unpredictable behavior.

Related to this example, Needham (2000, p. 69) observes that in the "ring of doubt", we can conclude the convergence of all points, some, or none of the points.

In the figure 9, we indicate a specific trajectory and analyze the regions that exhibit the behavior indicated in the figure below. In the blue trajectory we pointed the circular section. We described a closed route in a counterclockwise direction.

The length of the green vector must indicate if the total sums are smaller or bigger than radius.

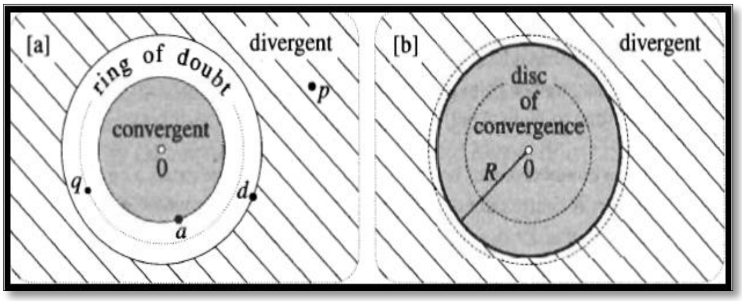

Figura 10. Needham (2000, p. 69) observe that exists a ring of doubt

Finally, we intended to bring some examples that to be unworkable when we neglect the technology. On the other hand, when we consider the historical scenario of the many mathematical concepts, we realize the important role of the figures and pictures that enable the transmission of heuristic ideas (HAIRER & WANNER, 08; EVES, 1969). We have shown in previous section, the employment of informal ideas for to verify specific problems (BOTTAZZINI, 2008). Recall the case of the series $\sum_{n\geq 1}^{\infty}\frac{(-1)^n}{\sqrt{n}}<\infty$ while the series $\sum_{n\geq 1}^{\infty}\frac{(-1)^n}{\sqrt{n}}\left(1+\sum_{n\geq 1}^{\infty}\frac{(-1)^n}{\sqrt{n}}\right)$ diverge. This was the counterexample provided by Dirichlet to Cauchy. Unfortunately, Cauchy was wrong in your conjecture commented by Bottazzini (2011). With the software, possibly, we can realize the meaning this counterexample. In addition, we highlight the relevant conceptual links indicated in the sequences, series and complex power series. The dynamic system Geogebra can improve the visualization of certain formal definitions, like we indicated by the figure 7. The student can independently infer the Cauchy´s conditions for series in IR and *C*, (DAVID & DONSINGM, 2010, p. 37).

Final remarks

Eves (1969, p. 379) notes that "Weierstrass was a very influential teacher, and his meticulously prepared lectures established an idea for many future mathematicians". From this point of view, we hope to influence the perspective and the approach of the mathematical teacher, regarding the use of the technology (ALVES, 2011). In our current context of the teaching, despite the influence of the "abstract generalization process", whose roots come from Weierstrass, we must consider the benefits of the technology and, in the specific way, the use of the software Geogebra in Analysis´ class.

We indicated qualitative properties of the series, the numerical behavior of the partial sums, with the aim to understanding the notion of convergence and divergence. In fact, thought the use of the software, we can visualize the relationships between the series and improper integral (integral-test). We can explore the neighborhood, with a determined radius of convergence and comprehend the existence of the "ring of doubt".

From the history point of view, Bloch (2011, p. 484-485) mentioning the pioneer work of Nicole Oresme (1323-1382), around 1350, he discussed the geometric series (see fig. 1). In this sense, Bloch (2011, p. 484) stressed that Oresme gave some pictures for prove some properties of the sum of twice Swineshead's series. Other mathematicians like Viète expressed intuitive ideas for the notion of convergence based upon the partial sums.

According to this author, Pietro Mengoli (1626-1647) obtained the sums

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1 \text{ e } \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{4}.$$

Among other things, "Mengoli deduced that is the sequence of partial sums of a series with positive terms is bounded then the series is convergent". (BLOCH, 2011, p. 484). The fact is that several properties related to the notion of real series can be studied in the Complex Analysis and in the teaching context.

With this scope, we brought in this article some examples that can be understood through visualization, perception and exploration of basic constructions with software. In fact, we discussed the understanding of the notion of convergence of power series in □ . With the objective of to understanding about the "ring of doubt" indicated in some books, we choice some possibilities of geometric interpretation by vector sums, like we showed in the figures 7, 8 and 9.

With the convenient and methodological use of this software Geogebra, we can explore several ideas pointed by Needham (2000). In this way, we can drag a mobile point (see figures 7 and 9) and verify some statements indicated in your book of Visual Complex Analysis. The qualitative properties discovered with this technological tool can overcome the logical meanings conditioned by mathematical symbols.

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