Drawing curves with the Dynamic System Geogebra: the Newton's method

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ABSTRACT: In this paper, we discuss some applications related to a systematic standard method in Algebraic Geometry. In fact, we bring a comparative discussion between old preoccupations related to describe a shape of a algebraic curve. Indeed, we find some authors that indicate and provide several static figures en virtue to guide the visual understanding. Actually, we can explore the potentialities of the Dynamic System Geogebra – DSG and compare, from a qualitative view, the constructions provided by this software.

KEYWORDS: Visualization, DSG, Newton´s Methdods, Algebraic Curves.

1 Introduction

In a general context, the graph behavior of certain classes of the mathematical objects is extremely relevant en virtue to acquire an intuitive understanding about its nature. In fact, when we dispose graphical representations of a object, we have the possibility to develop our discursive function related to a specific conceptual entity. In fact, when we visualize an object in 2D, we can realize a designation of a particular conceptual object. We can elaborate some statement and propositional sentences about the object. Moreover, we can act in a problem situation en virtue to develop some sequences of logical statement in a coherent way. Finally, as a human been, we can reflect about theses statement and its pragmatic values.

For example, what would we can declare about the following expression like $X^2 + XY + Y^2 + Y^3 + XY^3 + X^5 + X^2Y^3 + X^3Y^2 + XY^3 = 0$? (*) Moreover, can a particular procedure produce a graphical representation related to this complex planar equation? In a particular way, we have indicated in the figure 1, some steps related to the Newton's diagram. Its description can be finding in Vainsencher (2009, p. 40-41).

In a general manner, this procedure provides some criterion to describe, near at the origin, a trace of algebraic curve in the plane. However, when we lead of a complex case, like in the figure 1, become impossible to obtain entire information about the shape of the algebraic curve (*). In fact, below we determine some segments (in the left side) from the term $a_{ij}X^iY^j$ effectively present in the algebraic expression like $f(X,Y) = 0$.
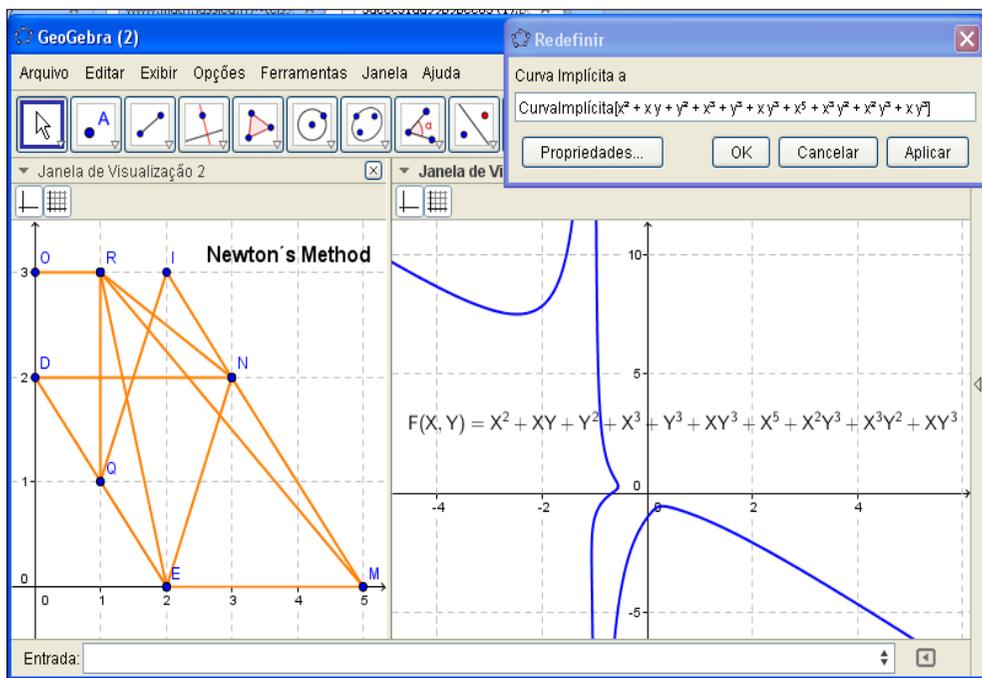
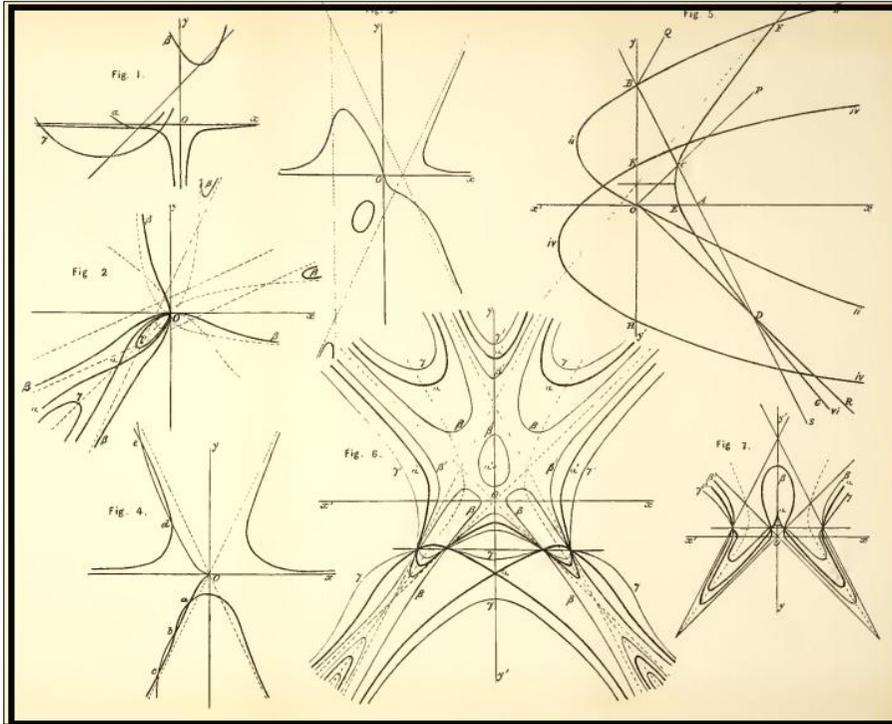**Figure 1.** Visualization of a algebraic plane curve with DSG

Well, in the next section, with the DSG´s help, we will bring some examples of the useful use en virtue the perception and the visualization. Moreover, this indispensable cognitive ability is a powerful tool en virtue the teaching and learning at the academic level (ALVES, 2014)

2 Visualizing with DSG´s help

In the standard literature, we find several methods related the description of the plane curves. In fact, Frost (1918, p. 167) explains that "at the same time, I have generally expect the student only to examine the figure, in other to see how the elements were combined.
 I shall now give a few rules for the systematic treatment of the equation of a curve, only recommending them as convenient plans to adopt, and leaving to the ingenuity of the student such modifications as particular forms of equations may suggest.".

Undoubtedly, we see the relevance attributed to the figures as an element that can to guide the understanding. In fact, in the figure 2, we show extraordinary figures provided in his old mathematical book.

Figure 2. Some curves described by Frost (1918, p. 167-168) and his systematic method

Contrary the static appearance that we see in the figure 2, we can visualize, with the DSG´s help, a dynamic representation of a algebraic curve. Clearly, its trace is less complicated. However, when we analyze its graph behavior (on the left side), we can conjecture a serious computational problem.

On the other hand. lets take the curve C described $x^6 - x^2 y^3 - y^5 = 0$. Given $t \in IR$, we will find the intersections points of C with the line $y = tx$. For this, we compute $x^6 - x^2 y^3 - y^5 = 0 \therefore x^6 - x^2 (tx)^3 - (tx)^5 = 0 \leftrightarrow x^6 - t^3 x^5 - t^5 x^5 = 0$. So, if we consider $x \neq 0 \therefore x^5 (x - t^3 - t^5) = 0 \leftrightarrow x = t^3 + t^5$.

Finally, we still obtain $y = tx = t^4 + t^6$. So, we obtain that $\alpha(t) = (t^3 + t^5, t^4 + t^6)$.

Now, we can obtain the velocity vector

$$\alpha'(t) = (3t^2 + 5t^4, 4t^3 + 6t^5) \therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3 + 6t^5}{3t^2 + 5t^4} \overset{t \neq 0}{=} \frac{t(4 + 6t^2)}{3 + 5t^2}.$$

So, in some works (ALVES, 2014), we can describe the trace's behavior from the velocity vector components or, the vector $\alpha''(t) = (x''(t), y''(t))$.
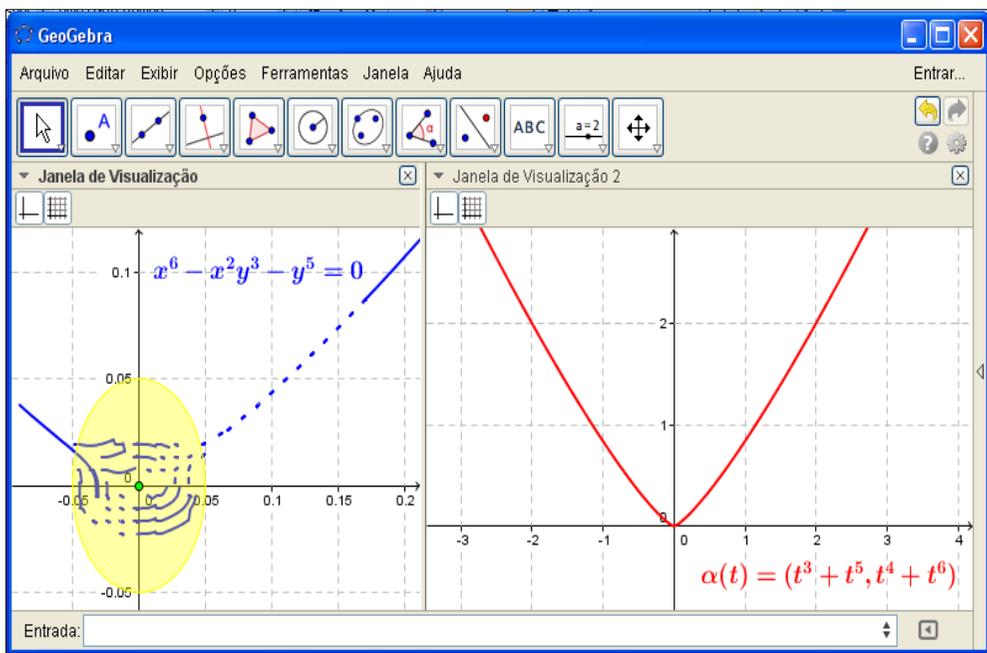
**Figure 3.** Visualizing some limitations of the DSG related to the trace of curves

In fact, from the expression $\dfrac{dy}{dx} = \dfrac{4t^3 + 6t^5}{3t^2 + 5t^4}$ if $t = 0 \leftrightarrow \dfrac{dy}{dx} = \dfrac{0}{0}$ and from this, we expect a cuspid point at the origin.

 Moreover, we can analyse the behavior of the components functions $(3t^2 + 5t^4, 4t^3 + 6t^5)$.

On the other hand, lets consider the instructions in Vainsencher (2009, p. 40-41). En virtue that, we choose the terms X^6, X^2Y^3, Y^5 and determine the following segments: $\overline{AB}, \overline{BC}, \overline{AC}$.

We determine the following parameter variation $0 \leq t \leq 1$ and, for which point $(6,0), (2,3), (0,5)$ we construct the parametric form related to these segments. For example, in the case $(2,3), (0,5) \therefore (1-t)(2,3) + t(0,5)$.

 From this expression, we indicate (on the right side), the pink trace corresponding to the $\overline{AB}$ variations. In the figure 4, we visualize and understanding how the geometric and graphical elements were combined in the dynamic construction.
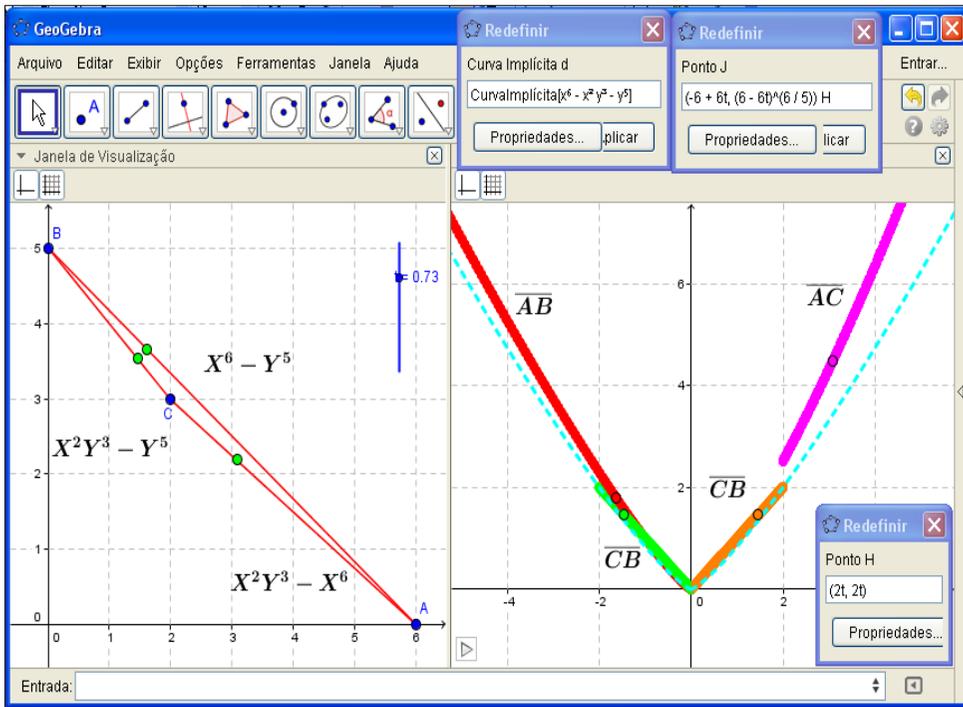
Figure 4. Description of the Newton's method with DSG´s help

Now, we can compare some perceptual data in the figures 3 and 4. How mentioned, we can consider some cases that it's become so difficult to employ the Newton´s methods. For example, we have considered

$$X^2 + XY + Y^2 + Y^3 + XY^3 + X^5 + X^2Y^3 + X^3Y^2 + XY^3.$$

In this case, we must take all expression like $a_{ij}X^iY^j$ with the condition $i \in \{1,2,3,5\}$ and $j \in \{1,2,3\}$ describe 18 (see fig. 1) segments and, for each of them, indicate the corresponding trace of curve. For example, in the case $(1,2)$ and $(1,3)$ we determine the segment $(1-t)(1,1)+t(1,3)$ corresponding $XY + XY^3 = 0 \therefore 1 + Y^2 = 0$.

Our last example is the curve $X^7 - X^2Y^4 - Y^6 = 0$. We visualize in the figure 5, a neighborhood at the origin $(0,0)$. On the left side we indicate its behavior in Cartesian coordinates, while in the right side, we have introduced in the equation $X^7 - X^2Y^4 - Y^6 = 0$

the substitution

$$Y = tX \therefore X^7 - X^2(tX)^4 - (tX)^6 = 0 \leftrightarrow X^7 - t^4X^6 - t^6X^6 = 0 \therefore X^7 = (t^4 + t^6)X^6$$

and under the condition $X \neq 0$ we obtain that $X = (t^4 + t^6)$ and $Y = (t^5 + t^7)$.

Finally, we can introduce the parameterization $\alpha(t) = (X(t), Y(t)) = ((t^4 + t^6), (t^5 + t^7))$. (see fig. 5)
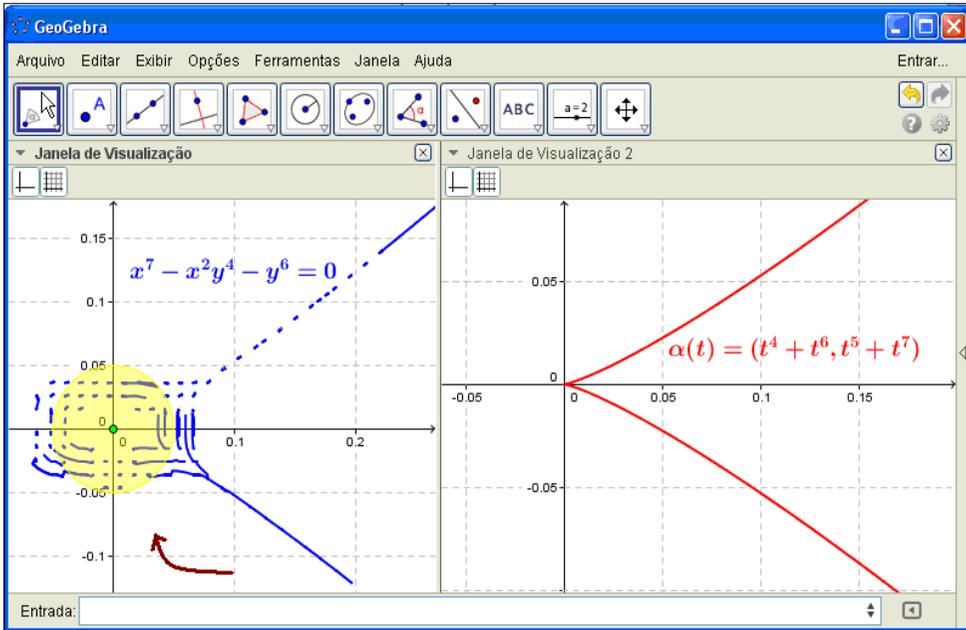
**Figure 5.** Visualizing some limitations of the DSG related to the trace of curves

In the figure 5 we visualize a strange behavior a this neighborhood centered at the $(0,0)$. In fact, in this point we have a singularity. We must observe that is really a mathematical computational problem obtain some correspondences between the Cartesians expression and a corresponding parameterization. Easily, in the case $X^3 + Y^3 - 3XY = 0$ is related to the $\alpha(t) = (x(t), y(t))$ in such way we show in the figure 6. On the other hand, if we take the following parameterization $\alpha(t) = (t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, t^{32})$, how to get an expression in the standard form $f(X,Y) = ?$ (another examples can be find in Albuquerque (1937) or Zdorov (1980))

We finalize with an example of Vainsencher (2009, p. 41). In the figure 7, we can compare the dynamic construction provided by DSG with the static figure suggested by this author in his Brazilian classical book. Once again, with the help of software, we can establish a visual correspondence relating to each part of the trace of the curve with the segment corresponding to the left side. From this dynamic construction, we have to observe that there are some peaces of the figure that is not in a desired correspondence.

Moreover, Vainsencher (2009, p. 42) explains why this methods works, when declares that "without going into details, the method works because each segment of the diagram selects terms of the equation are infinitesimals of the same order, the other points in the opposite half-plane to the origin representing higher-order terms.".
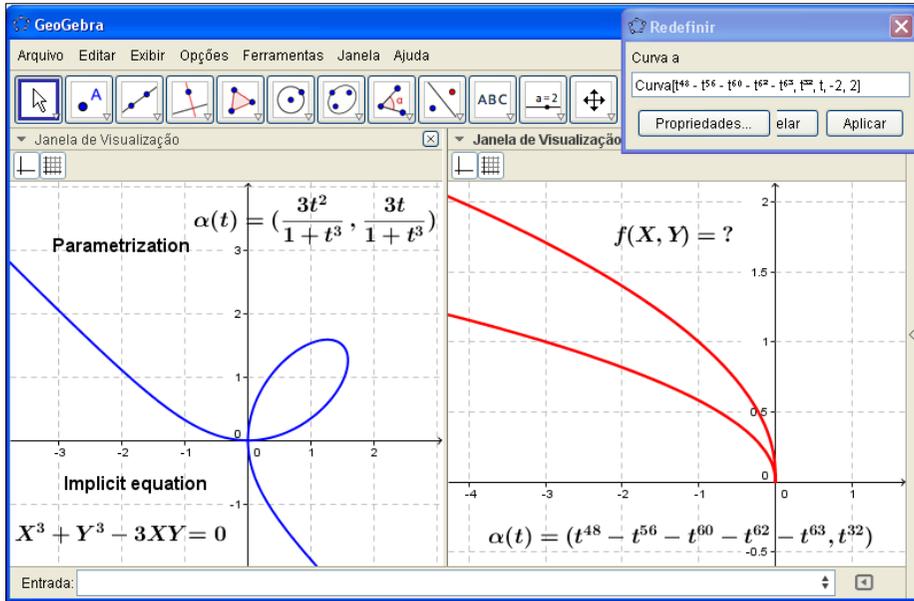
Figure 6. Some correspondences between the Cartesians equation and the parameterization provided by the DSG
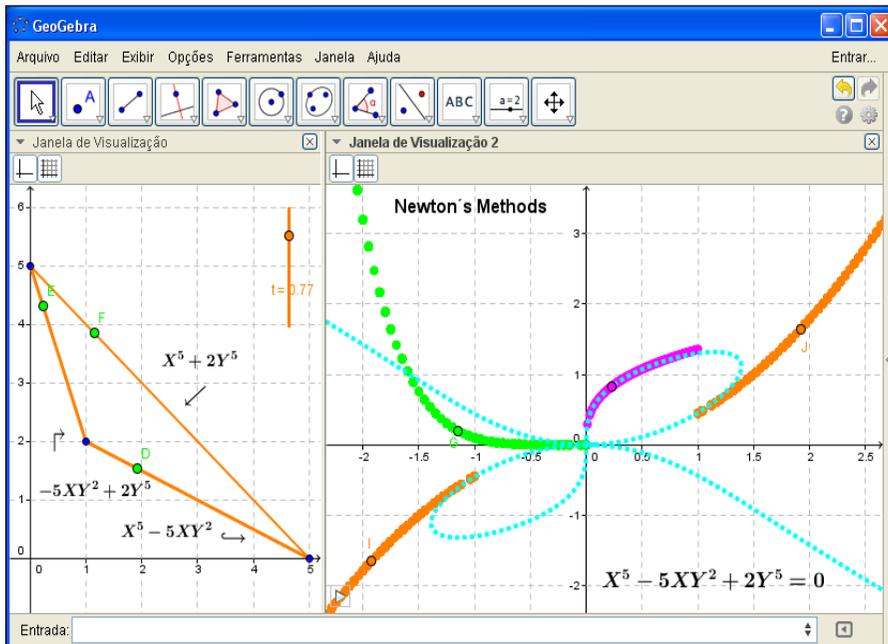
Figure 7. Vainsencher (2009, p. 41) describe some steps relatively the Newton's methods

In the figure 7, we determine the points $(5,0),(1,2),(0,5)$ and from these, we construct the tree corresponding segments $\left(\text{condition } 0 \le t \le 1 \right)$:

$$[(1-t)(5,0)+t(1,2)],[(1-t)(1,2)+t(0,5)],[(1-t)(5,0)+t(0,5)].$$

Finally, in the right side, we obtain a visual animation the several graphical-geometric elements. Certainly, in the visual and perceptual way, we find certain limitations that can be fulfilled throughout the actual technology. Still in the figure 7, on the right side, we have indicated the final behavior related to the curve

$$f(X,Y) = X^5 - 5XY^2 + 2Y^5 = 0.$$

In color blue, we indicate this correspondently graph behavior and compare the final result. In this case, we perceive some peaces of graph that are not in a visual correspondence.

Final remarks

In this paper, we bring to discussion about a standard construction of the algebraic plane curves. In fact, we have discussed some applications of the Newton´s method. We find always some author's instruction about this tool for the describing a shape and the graphical behavior of an algebraic curve trace in the form $f(X,Y) = 0$.

Although, we have to observe some limitations related to own method and the DSG´s use.

In fact, when we lead with simple classical curves is possible the method's applications. However, in some cases and en virtue the great quantity of terms $a_{ij}X^iY^j$ is unworkable the effective description. Moreover, we have showed some unexpected example of Cartesian equation en virtue the graph behavior provided by this software. (see fig. 3 and 5).

Relatively the Newton's methods, we find some authors (ALBUQUERQUE, 1937; FROST, 1918; ZWIKKER 1950) that describe it through drawings and some figures. This is an important effort en virtue a heuristic understanding, however, in certain specific topics, is almost unworkable imagine a correspondently figure (see figure 2). In fact, we indicate to the careful reader examine some example Albuquerque (1937, p. 20).

Finally, from the exploration of DSG, we structured a preliminary approach to some introductory topics in Algebraic Geometry. In fact, this mathematical branch is just an example that we can work out with a view to the teaching, learning and a better understanding of the academic student. In fact, in the figure 8 we can visualize the author's effort en virtue to transmit his mathematical ideas by means of figures.
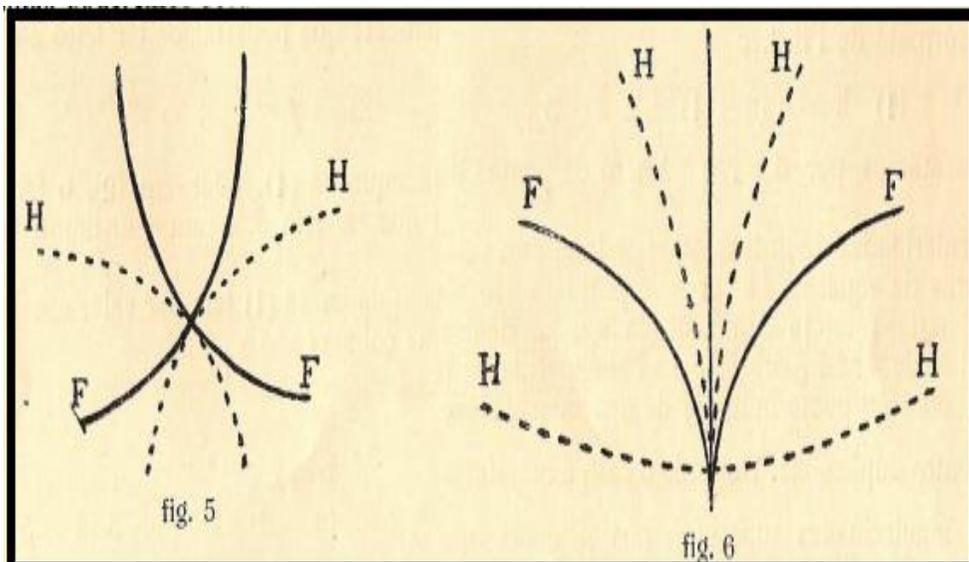
Figure 8. Albuquerque (1937, p. 30-31) indicates a heuristic and visual interpretation for some mathematical properties in Algebraic Geometry

Our view is based on the value of visualization as a tool for approximation and stimulation of math students. Through a didactical and structured approach, that approximates to the spirit of professional mathematicians in the past centuries. We can see something similar when we read the book of Vainsencher (2009, p. 40, p. 95). Indeed, this author makes several references to the term "by intuition" or "following by intuition". Your perspective can be extended in order to benefit other mathematical students who do not have a very thorough knowledge of Algebraic Geometry. In this way, the DSG provide a perfect scenario for a heuristic learning.

References

[AJ37]   **J. R, Albuquerque** – *Geometria e Análise: teoria geral das curvas algébricas e complementos do cálculo integral*, Lisboa: Editora Universitária,1937.

[FR14]   **F. R. V. Alves** – *Semiotic Register and the Internal Transition to Calculus: Elements for a Didactic Engineer*, Lima: Catholic University of Peru, Conference, 2014.

[PF18]   **P. Frost** - *An elementary treatise on Curve Tracing*, fourth edition, London: Mcmillan and Co. 1918.

[PBP00]   **P. Beelen & R. Pellikan** – *The Newton Polygon of plane curves with many rational points*, Designs, Codes and Cryptography, vol. 21, p. 41-67, 2000.

[VI09]   **I**. **Vainsencher**– *Introdução às Curvas Algébricas*, Rio de Janeiro: IMPA, 2009.

[VI09]   **Yu. A**. **Zdorov** – *Remarkable curves*, Moscow: MIR, 1980.

[VI09]   **C. Zwikker** – *The advanced geometry of Plane Curves and their application*, New York: DOVER Publication, 1950.