Constructive Identification Of An Ellipse Using GeoGebra

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Abstract : "The Identification of an ellipse" by definition refers to the process of getting its equation, its center , the lengths of the semi principal axes, the location of its two foci, and the two directrices. When the picture alone is drawn these details may not be available . Under these circumstances, pure geometric constructive methods exist that can be used to identify the above features. The discussion is made more interesting by assuming the following situation. With the help of the free geometric software GeoGebra an ellipse is drawn, using a typical second degree equation in the variables x *and* y . The second degree equation is suppressed initially. Through the geometric constructive methods, the relevant details are obtained. In a later section the second degree equation is brought in. From the equation, the necessary information regarding ellipse will be presented based on theoretical results. The results are then compared to see how effective the geometric constructive methods are.

Key words: Auxiliary circle, directrices, foci and principal axes of an ellipse.

MSC subject classification number: 11R05.

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1. INTRRODUCTION:

Many curves in Euclidean plane geometry can be constructed by using mechanical devices. The book "Practical Conic Sections" authored by J. W. Downs [Downs93] , in chapter one gives nine methods for constructing an ellipse. Daniel Scher in his book, "Exploring conic sections with The Geometer's Sketchpad" [Scher95] gives very many interesting constructions of the ellipse. It is not difficult to imagine an ellipse drawn in a rectangular sheet of paper so that the resulting ellipse appears to be tilted. Except for the figure, "the identification of the ellipse" is not there. For simplicity, such an environment can be easily created by taking a second degree polynomial equation in two variables x *and* y and then create the ellipse using the free geometric software GeoGebra. When the ellipse is printed out in a rectangular sheet of paper, it will only show the ellipse and not the equation that generated it. How does one identify all the details regarding the ellipse? . This article gives a constructive procedure all with the help of the free software GeoGebra that provides "the identification of the ellipse".

The results obtained from the constructive method are compared with the results obtained from theoretical calculations based on the equation that was used to create the ellipse by the software GeoGebra.

2.The raw ellipse:

The figure that is being shown is based on a second degree equation that will be revealed later .

Here the diagram is presented.

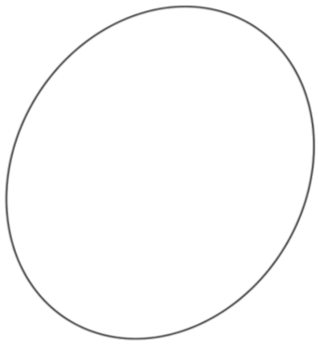

Figure 1

The diagram does not have the details that constitute “the identification of the ellipse”. The process of obtaining the above information is dubbed by the author as “The constructive identification of the ellipse”. This can be done using the tools that are provided in a “Geometry box” that students carry in high schools. However, the free geometric software GeoGebra makes the above construction fairly easy to achieve. This constructive procedure is given in the next section.

3. The constructive identification of the shown ellipse

Many of the concepts associated with an ellipse have their origins that are related to its constructible structure of the ellipse, as well as the physical reality behind it. For instance if the principal axes are already drawn on an ellipse in a sheet of paper, and the ellipse is cut out from the paper, there are two perfect foldings of the cut out ellipse only along the two principal axes so that the folded portion lies perfectly synchronized with the portion of the ellipse that is beneath. This is the only physical way that one can check, whether the claimed principal axes are indeed what they are professed to be. Further if both these foldings are done one along the principal minor axis and the other along the principal major axis and creases are on the lines the meeting point of these two creases is precisely the center of the ellipse. Unfortunately, this recognition alone is not enough. The identification of the center precedes other things and this is the first stage of the construction.

A diameter of an ellipse is defined as a finite length segment that is the locus of all mid-points of parallel chords that are drawn on an ellipse in a given direction. The line joining two mid-points of two parallel chords of an ellipse essentially determines one unique diameter, when the above line intersects the ellipse in two points. Every diameter of an ellipse is a finite length segment that goes through the center of the ellipse such that the center of the ellipse is also the mid point of each diameter. Normally text books recommend the creation of two diameters that intersect each other. The intersection of two such diameters is the center of the ellipse. In our context two diameters are unnecessary, since GeoGebra can easily locate the mid-point of any single diameter.

The following result is very crucial to the construction of the principal axes of the ellipse. It is given next.

Theorem 3.1 *Let* K *denote the center of an ellipse* c *. Let* P *denote a moving point on the ellipse* c*. With the point* K *as center let a varying circle be drawn with radius* $r = KP$*. As* P *moves on the ellipse, the moving circle intersects the ellipse always in four points, except on four occasions . When* P *coincides with one of the two end points of its minor axis the created circle touches the ellipse from inside. When* P *coincides with one of the two end points of its major axis, the created circle is known as the Auxiliary circle, and the ellipse touches the Auxiliary circle from inside. If any circle is constructed that has center* K *and radius* $r = KP$ *that has four points of intersection with the ellipse, the cyclic quadrilateral formed by the four points of intersection of the ellipse with the circle is a rectangle. The diagonals of the ensuing rectangle constitute two diameters of the ellipse (also diameters of the circle). Further the lines that are drawn through the mid-points of opposite parallel sides of the rectangle determine the two respective principal axes of the ellipse. The principal axes are precisely the two diameters of the ellipse determined by the lines on the ellipse through pairs of mid-points of opposite parallel sides of any such rectangle that can be created by the four points of intersection of the circle with the ellipse.*

To make the constructive identification of the ellipse very simple, using GeoGebra, three points $A, B, and\ C$ are selected on the ellipse using "point on the object" command. A varying chord AB is created. Both the points $A\ or\ B$ can be moved arbitrarily on the ellipse. Through the point C a parallel line is drawn to the chord AB meeting the ellipse at the points $C\ and\ D$. The mid-points of these parallel chords are denoted by $E\ and\ F$. Since the figure is visible in the printed sheet or in the computer monitor, it is quite easy to create a specific diameter GH through the mid-points $E\ and\ F$ of the parallel chords $AB\ and\ CD$ in such a way that the circle with K as center and radius

$$r = KG = KH = \frac{GH}{2}$$

intersects the ellipse in four points that are common to the circle and the ellipse. Two of those four points are already the end points $G\ and\ H$ of the common diameter GH of the circle and ellipse. The remaining two points are labeled as $L\ and\ M$. The possible maneuvering of the chords $AB\ and\ CD$ makes the above construction possible. The cyclic quadrilateral $GLHM$ through the four points $G, L, H, and\ M$ is a rectangle shared by the ellipse and the circle. The diagonals of this particular rectangle are themselves diameters of both the ellipse and the circle. The perpendicular bisector of two non parallel sides $GL\ and\ LH$ of the rectangle $GLHM$ creates the two principal axes of the ellipse, when these perpendicular bisectors intersect the ellipse in four points $P, Q, R, and\ S$. Of course these perpendicular bisectors go through the mid-points of the parallel sides of the rectangle. The smaller line segment between $PR\ and\ QS$ would be the minor axis, and the larger line segment would be the major axis. After drawing the principal axes with the help of GeoGebra, a cutting can be made of the new ellipse in a sheet of paper. The folding test can be made to confirm that they are indeed the principal axes. The logic behind this construction is based on **Theorem 3.1.**

The posted Figure 2 represents the culmination of the first two stages of construction.

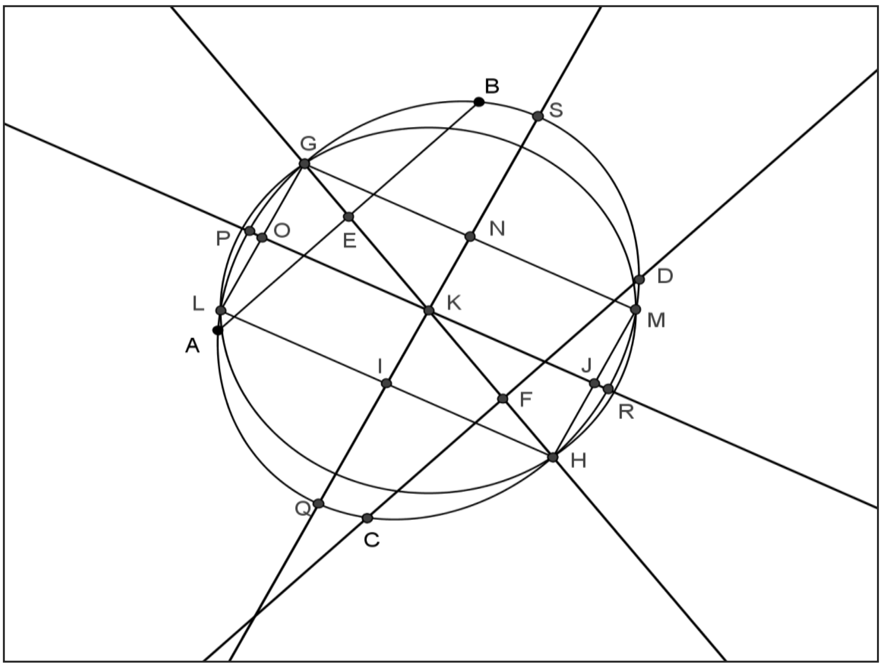

Figure 2

The third stage of the construction involves the construction of the two foci and the construction of the two directrices of the ellipse. The final figure needs to convey only the essentials and some important data. With that in mind things that clutter the final figure are removed from the final figure by using the commands "Hide the object" and "Hide the label" from the Edit menu. One by one all things that clutter the final diagram can be removed. After removing unnecessary lines from the previous Figure 2, it is clear that the axis $PR\ and\ QS$ are respectively the minor axis, and major axis of the ellipse. The remaining lines and unnecessary points are kept away from the monitor screen by using the "Hide object" button. With that as the set up, the vertex P is chosen as the center for a circle, with radius $r_1 = KQ = semi - major\ axis\ length$ as its radius. The two points of intersection that lie inside the major axis QR are labeled $V\ and\ W$. $V\ and\ W$ are the two foci of the ellipse. The Cartesian coordinates of the four vertices, and the two foci are calculated by GeoGebra, when once these points are created using various menus. The Auxiliary circle of the ellipse is drawn with center $K\ and\ radius = KQ$. This circle in the third Figure 3, is labeled by the letter q. The ellipse is labeled c. The polar line of the focus V with respect to the Auxiliary circle q is the directrix denoted by the label l. The polar line of the focus W with respect to the Auxiliary circle q is the second directix of the ellipse and it is labeled m. The points of intersection of the two directrices with the extensions of the major axis are the two feet of the directrices labeled $T\ and\ U$. The final diagram is posted next. It includes all the relevant information.

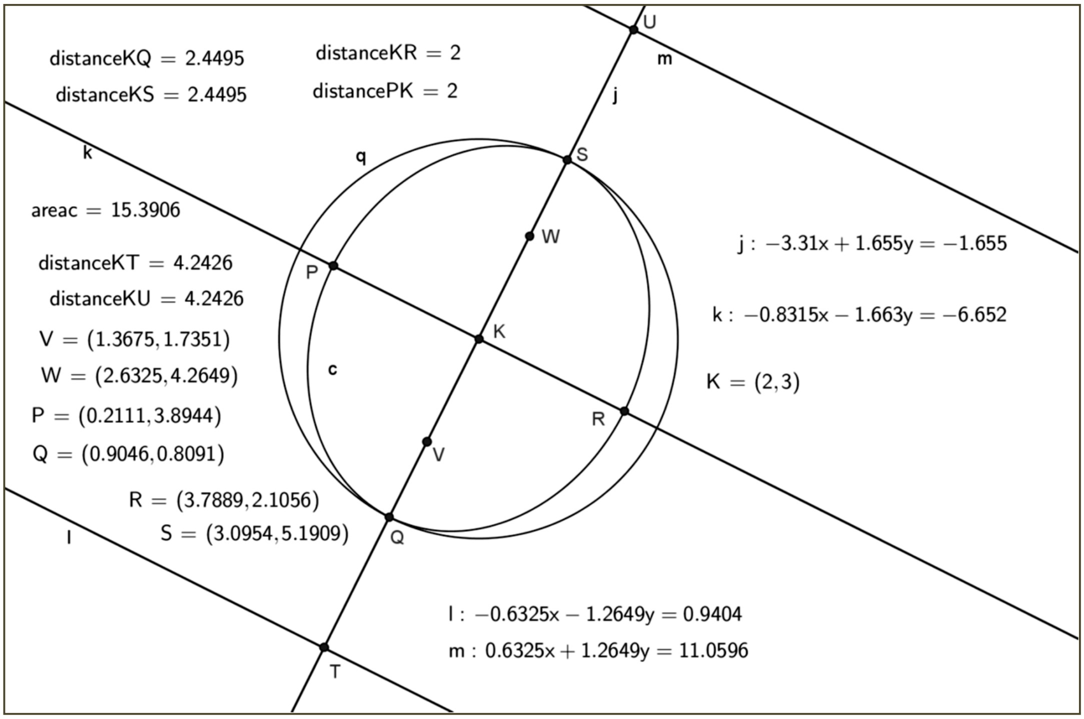

Figure 3

Final summary: $K = (2,3)$ is the center of the ellipse. The vertices of the ellipse are given by

$$P \cong (0.2111, 3.8944), Q \cong (0.9046, 0.8091), R \cong (3.7889, 2.1056), S \cong (3.0954, 5.1909) \quad (3.1)$$

Semi-major axis length is given by $a = KQ = KS \cong 2.4495 \ cm$. Semi-minor axis length is given by $b = KP = KR \cong 2 \ cm$.

Area of the ellipse as given by GeoGebra is $\cong 15.0936 \ sq.cm.$ The calculator value of $\pi \ a \ b \cong 15.0936 \ sq.cm.$ rounded to four decimal places. The two foci are given by

$$V \cong (1.3775, 1.7351) \ and \ W \cong (2.6325, 4.2649) \qquad (3.2)$$

Equations of the major axis line and minor axis line are given by GeoGebra in the form

$$j: -3.31\,x + 1.655\,y = -1.655, \qquad k: -0.8135\,x - 1.663\,y = -6.652 \qquad (3.3)$$

The equations of the two directrices as given by GeoGebra are

$$l: -0.6325\,x - 1.2649\,y = 0.9404, and\ m: 0.6325\,x + 1.2649\,y = 11.0596 \quad (3.4)$$

The feet of the two directrices as given by GeoGebra are

$$T \cong (3.8974, 6.7947) \ and \ U \cong (0.1026, -0.7947) \qquad (3.5)$$

Remark 3.1 It can be shown easily that the Auxiliary circle meets the extension line of the minor axis at two points one above the point P, and the other below the point R. If these points are labeled $P_1 \ and \ R_1$ two tangents can be drawn to the ellipse from P_1. These two tangents intersect the major axis line precisely at the two feet

$$T \cong (3.8974, 6.7947) \ and \ U \cong (0.1026, -0.7947) \qquad (3.5)$$

From the above information, the two directrices can be constructed as the lines perpendicular to the extension line of the major axis at $T \ and \ U$. Similarly it can be shown that the two tangent lines drawn to the ellipse at R_1 go through the same feet $T \ and \ U$. The quadrilateral P_1TR_1U circumscribes the given ellipse and indeed it is a rhombus. The side length of the rhombus P_1TR_1U is approximately $4.899 \ cm$.

4. Theoretical determination of the ellipse

The raw ellipse in Figure 1 was drawn using GeoGebra by feeding the exact equation

$$\Phi(x,y) \equiv 14\,x^2 - 4\,x\,y + 11\,y^2 - 44\,x - 58\,y + 71 = 0 \qquad (4.1)$$

in the input box. With (4.1) as the basis, the theoretical center is the result of solving the two linear equations

$$\frac{\partial \Phi(x,y)}{\partial x} = 0 \ and \ \frac{\partial \Phi(x,y)}{\partial y} = 0 \qquad (4.2)$$

The equations are thus

$$28\,x - 4\,y = 44 \quad and \quad -4\,x + 22\,y = 58 \qquad (4.3)$$

It should be mentioned that the two equations in (4.3) *also provide two diameters of the ellipse, when the lines are drawn by GeoGebra by taking their points of intersection with the ellipse.* The unique solution of the two equations is the point $K = (2,3)$. This is in conformity with the practical construction given by previous section. Let

$$h = 2 \;\; and \; k = 3 \qquad (4.4)$$

By transferring the origin to the center $K = (h,k) = (2,3)$ with respect to the parallel axes through the point $K = (h,k) = (2,3)$ the equation (4.1) gets transformed into the new equation

$$14\,x^2 - 4\,x\,y + 11\,y^2 = 60 \qquad (4.5)$$

The above equation (4.5) can be rewritten as

$$\frac{7}{30}\;x^2 - \frac{1}{15}\;x\,y + \frac{11}{60}\;y^2 = 1 \qquad (4.6)$$

The equation (4.6) is sufficient to determine the lengths of the semi-major axis and semi-minor axis. The next result that is presented is a **Theorem** that is given in the book “Elements of Coordinate Geometry” authored by S. L. Loney [Loney12]. *The article 364, pages 338 to 341 contains the relevant information including the example that is used in this paper, in this section. The Corollaries that are stated in page 340 are the ones that will be used in this section.*

Theorem 4.1 *Let the equation of a central conic section be of the form*

$$a\,x^2 + \; 2\,h\,x\,y + b\,y^2 = 1 \qquad (4.7)$$

If the conic section represents an ellipse, then the reciprocals of the squares of the semi-axes are the roots of the quadratic equation

$$Z^2 - (a+b)\,Z + (ab - h^2) = 0 \qquad (4.8)$$

The area of the ellipse is then given by

$$Area\ of\ the\ ellipse = \frac{\pi}{\sqrt{((ab - h^2))}} \quad sq.units. \qquad (4.9)$$

Remark 4.1 The central conic (4.7) represents an ellipse if $(ab - h^2) > 0$ or equivalently

$h^2 < a\,b$. If $\alpha > 0$ *and* $\beta > 0$ are the semi-lengths of the principal axes of the ellipse **Theorem 4.1** implies the following:

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = a + b \quad and \quad \frac{1}{\alpha^2\,\beta^2} = (ab - h^2) \quad (4.10)$$

In the context of the example $a = \frac{7}{30}$, $h = -\frac{1}{30}$ and $b = \frac{11}{60}$. Clearly $h^2 < a\, b$ is satisfied. The conic section is clearly an ellipse, which is also seen from the figure. The relations (4.10) yield

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{5}{12} \quad and \quad \frac{1}{\alpha^2\, \beta^2} = \frac{77}{1800} - \frac{1}{900} = \frac{75}{1800} = \frac{1}{24} \qquad (4.11)$$

It follows from (4.11) that

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{5}{12} \quad and \quad \frac{1}{\alpha^2} - \frac{1}{\beta^2} = \pm\, \frac{1}{12} \qquad (4.12)$$

Two solutions are possible for

$$\frac{1}{\alpha^2} \; and \; \frac{1}{\beta^2} \qquad (4.13)$$

They are given by

$$\frac{1}{\alpha^2} = \frac{1}{4} \quad or \quad \frac{1}{\alpha^2} = \frac{1}{6} \qquad (4.14)$$

$$\frac{1}{\beta^2} = \frac{1}{4} \quad or \quad \frac{1}{\beta^2} = \frac{1}{6} \qquad (4.15)$$

Thus

$$\alpha^2 = 4 \;\; or \;\; \alpha^2 = 6 \;\; and\; similarly \;\; \beta^2 = 4 \;\; or \;\; \beta^2 = 6 \qquad (4.16)$$

The pair $\alpha^2 = 6$ and $\beta^2 = 4$ gives one set of principal axes. In this set up the semi-major axis length is given by $\alpha = \sqrt{6} \;\; \cong 2.4495\, units$. The length of the semi-minor axis is given by $\beta = 2\, units$. The second solution has the values of α and β reversed. To understand this ambiguity, the angle of rotation for the equation (4.6) has to be taken into consideration. This is given by

$$\tan(2\,\theta) = \frac{2\,h}{(a-b)} = -\,\frac{4}{3} \qquad (4.17)$$

However

$$\tan(2\,\theta) = \frac{2\ tan(\theta)}{(1-(\tan\theta)^2)} = -\ \frac{4}{3} \qquad (4.18)$$

Thus either $\tan\theta = 2\ \ or\ \ \tan\vartheta = -\ \frac{1}{2}$. This means that the directions of the principal axes provided by the second solution is obtained by the rotation of the directions of the principal axes provided by the first solution through an additional rotation by $\frac{\pi}{2}\ radians.$

Equation (4.5) can be put in the polar format

$$r^2 = 60\left\{\frac{1+\ (\tan\theta)^2}{14-4\ \tan\theta + 11\ (\tan\theta)^2}\right\} \qquad (4.19)$$

The substitution of the choice $\tan\theta = 2$ gives $r^2 = 6\ \ or\, r = \sqrt{6} \cong 2.4495\, units.$ The substitution of $\tan\vartheta = -0.5$ gives $r^2 = 4.$ The identification that was provided by GeoGebra in Figure 3 corresponds to $\tan\theta = 2$. In this case semi-major axis length is $\sqrt{6} \cong 2.4495\ cm.$ The semi-minor axis length is $2\ units.$ The exact area of the ellipse is $2\,\pi\,\sqrt{6}\ \ sq.cm. \cong 15.3906\ sq.cm.$ All these details are in agreement with the constructive identification of the ellipse provided by GeoGebra. The slopes provided by principal axes lines are given by $\tan\theta = 2\ \ and\ \ \tan\vartheta = -\ \frac{1}{2}$ respectively in Figure 3.

5. CONCLUSIONS

The word “Geometry” originally meant measuring the earth. Before Euclid formulated his well known “Euclidean Geometry” constructive methods based on using mechanical devices existed. *The workers that built the pyramids of Egypt, buildings like the Pantheon in Rome, and temples around the world knew how to build these architectural wonders. May be Euclid formulated his Euclidean Geometery to explain the constructive experimentation that went on for centuries before his arrival. These architectural constructions preceded the development of Euclid’s Geometry. Many of the seating arrangements in the sports arenas are located on ellipses . This can be seen in Roman architecture scattered in Europe. While there is undeniable beauty involved in theoretical branches of Geometry, one also cannot forget the experimentation that went on for centuries based on mechanical tools. The purpose of this article is one of bringing back this experimentation based on mechanical tools. When the ellipse is drawn by mechanical means the important details regarding the “identification of the ellipse” is entirely possible without Coordinate Geometry. However if the ellipse is drawn based on a second degree polynomial equation in* x *and* y *, both the theoretical and constructive methods yield nearly the same conclusions.*

Acknowledgment: *The author is grateful to the creators of the free GeoGebra software, without which this work would have been impossible.*

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