INEQUALITIES IN THE HISTORY OF MATHEMATICS: FROM PECULIARITIES TO A HARD DISCIPLINE

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ABSTRACT: This theoretical contribution comes from a broader study that investigates undergraduate students' conceptions of inequalities. History of inequalities is looked into in a search for an answer to the question: Why are inequalities hard to meaningfully manipulate and understand? Memorable dates in the development of inequalities and the symbols for representing inequalities are highlighted. Implications for the teaching of mathematics are identified.

KEYWORDS: Inequalities; History of Inequalities; Epistemological Obstacles.

INTRODUCTION

It is well documented in educational studies that students encounter problems when manipulating inequalities or when interpreting what an inequality is and what does a solution of an inequality represent (Linchevski & Sfard, 1991; Bazzini & Tsamir, 2004). Many studies on inequalities reported mostly on students' misconceptions of inequalities (Linchevski & Sfard, 1991; Bazzini & Tsamir, 2004; Tsamir, Tirosh, & Tiano, 2004; Sackur, 2004). Some attempts were also made to decipher the nature of those misconceptions (Linchevski & Sfard, 1991; Tall, 2004; Dreyfus & Hoch, 2004).

The investigation into the history of inequalities presented here attempts a shift from *how students perform on inequalities* to *why students perform poorly on inequalities* and provides some answers in this regard.

1. WHY ARE MATHEMATICS EDUCATORS LOOKING AT THE HISTORY OF A CONCEPT?

When the teaching, learning, or understanding of a concept encounters problems, there is a tradition in mathematics education research to turn the search for the answer to the problem toward the history of the concept (Cornu, 1991). In the development of the concept, one may find information about periods of slow development. There could be an indication somewhere that the concept had been creating problems to mathematicians first. As it is well known, Hippasus probably died for discovering irrational numbers. Even if the Greek mathematicians had previously experienced incommensurability, they had problems accepting that the world is not fully explained by whole numbers and the relationship between them (Kline, 1972). Such incidents inform us about epistemological obstacles associated with irrational numbers.

Epistemological obstacles are a "way to interpret some of the recurrent and non-aleatorical mistakes that students make when they learn a specific topic" (Radford, 1997, p.8). Radford follows Brousseau's classification of recurrent mistakes in order to have a clear picture of epistemological obstacles:

(1) an *ontogenetic source* (related to the students' own cognitive capacities, according to their development);

(2) a *didactic source* (related to the teaching choices);

(3) an *epistemological source* (related to knowledge itself) (Radford, 1997, p.9).

Epistemological obstacles could be identified for a concept under observation by confronting the obstacles found in historical texts with the recurring mistakes our students are making when working on that concept. Teaching a concept linked to epistemological obstacles and being aware of that, the educator could plan when and how it would be more appropriate to introduce that concept to the students to avoid unnecessary hardship. Also, the teacher can work on training the students to practice that concept even when the meaning is obscured and manipulation mistakes are persistent (Sfard, 1995). Reaching procedural fluency could be a step toward relational understanding.

Several studies indicate that the history of mathematics is valued not for detecting epistemological obstacles only, but for informing the teaching of mathematics as well as research in mathematics education in various other ways (Bagni, 2005; Burn, 2005). Historical anecdotes used by many teachers as motivators for the study of a concept are viewed as a naïve but powerful approach of history to didactics (Radford, 1997). Recapitulation, or presenting topics through their historical development, is another way of using the history of mathematics in class, since the method essentially sets the stage for the students to recreate the concept (Radford, 1997).

Moreover, similarities between the historical development of a concept and its cognitive growth have been observed (Bagni, 2005). For example, Harper (1987) used the history of algebra to predict young students' intuitive algebraic thinking. The task was to solve a Diophantine problem requiring students to find two numbers when their sum and their difference were known. Nearly half of his participants used rhetorical or syncopated algebraic arguments to solve the problem. The Vietan symbolic algebra was the least popular method for solving the problem, since the students were unable to use the parametric setting to obtain the general solution. The argument here is that, historically, from rhetorical to symbolic algebra, there were centuries of new accumulations in mathematics. In parallel, didactically, students experience a huge jump in understanding and manipulating algebraic concepts when going from simple equations to parametric ones (Sfard, 1995).

Burn (2005) argues that the historical development of a mathematical concept "can reveal actual steps of success in learning" (p.271) that concept. This exploration could be applied when the research in education reveals that the understanding of a concept is not "consonant with students' intuitions" (p.271), which seems to be the case with inequalities (Bazzini & Tsamir, 2003). Burn uses the history of mathematics to fill the gap between a "pre-Zeno mindset to Weierstrassian viewpoint" (p.271) relative to the limits concept. The history of a mathematical concept has been successfully used as a longitudinal study of learning. Exposing his students to the classical Greek mathematicians' use of inequalities, in a pre-real numbers context, to help them see the power of inequalities in a more intuitive and, at the same time, rigorous way, Burn (2005) argues that the understanding of the rigorous proof of limits improved. For the purpose of my study, Burn's findings that "[p]roofs of equality by means of inequalities precede the notion of limit" (Burn, 2005, p.1) are twofold: (1) they motivate my search into the history of mathematics for informing teaching and (2) they provide leads for further digging for inequalities in the history of mathematics.

2. THE EVOLUTION OF THE INEQUALITY CONCEPT

As a discipline of study, inequalities do not have a long history. As a mathematical concept, however, they were not foreign at all to ancient mathematicians (Bagni, 2005). The ancients knew "the triangle inequality as a geometric fact" (Fink, 2000, p.120). They were also aware of the arithmetic-geometric mean inequality, as well as the "isoperimetric inequality in the plane" (Fink, 2000). Euclid used words 'alike exceed', 'alike fall short' or 'alike are in excess of' to compare magnitudes (Kline, 1972, p.69). The definition, "The greater is a **multiple** of the less when it is measured by the less," (Katz, 2009, p.74) shows that the mathematicians of ancient times were adept at comparing magnitudes and expressing the relationship between them.

Inequalities have been assisting mathematical discoveries from Classical Greek Geometry to Modern Calculus and it took two millennia to change the status of inequalities from mere support for some mathematics to *Inequalities* as a discipline of study (Fink, 2000). Today, there are two journals of inequalities – *The Journal of Inequalities and Applications*[1] and *The Journal of Inequalities in Pure and Applied Mathematics*[2] – as well as many other mathematics publications that print papers with the "sole purpose […] to prove an inequality" (Fink, 2000, p.118). The path that inequalities followed from Antiquity to the end of the second millennium is investigated in the following sections.

2.1 Inequalities in Antiquity

In his *Treatise of Algebra* (1685) Wallis acknowledged his debt to Euclid in relation to limiting arguments. (Burn, 2005, p.291)

Greek mathematicians were profoundly aware of "the power of inequalities to obtain equality" (Burn, 2005, p. 271). Burn employs a metaphor – called "the vice" – to describe the properties of inequalities that Ancient mathematicians used to help produce equality. The vice is similar to what we call the squeeze rule in calculus. It consists of the following argument: when a number A is squeezed in between two small quantities, $-\varepsilon < A < \varepsilon$,

1 Journal of Inequalities and Applications (JIA) issued its first volume in 1997. JIA is a multi-disciplinary forum of discussion in mathematics and its applications in which inequalities are highlighted.

2 Journal of Inequalities in Pure and Applied Mathematics (JIPAM), founded in 1999 by the Victoria University in Australia members of the Research Group in Mathematical Inequalities and Applications (RGMIA).

for all positive numbers ε, then the number $A = 0$. The inequality $-\varepsilon < A < \varepsilon$ works as a carpenter's vice, compressing the inner quantity as much as to leave room for only one number in between $\pm\varepsilon$, and that number is zero. The vice could be used to squeeze a difference of two numbers: $-\varepsilon < A - c < \varepsilon$, and to prove the equality $A = c$ (Burn, 2005).

Greek mathematicians were not aware of the existence of negative numbers. However, they seemed to have used the vice. Here is an account of the vice in Euclid's Elements:

> Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proven even if the parts subtracted are halves. (Katz, 2009, p.82)

The proposition states that by taking a small quantity, compared to a bigger one ($\varepsilon < B$), one can get a smaller quantity than the smaller of the two initial quantities by successively subtracting halves from the big one. Thus, the inequality becomes $B/2^n < \varepsilon$. This can be turned into $B < 2^n\varepsilon$, an inequality which signifies that a multiple of ε exceeds B (Burn, 2005). Classical Greeks as well as Archimedes used the potential of this inequality to calculate the volume of a pyramid as one third of the area of the base times the height (Burn, 2005), and many other similar results.

Archimedes also used the method of exhaustion to calculate many important results on areas and volumes. The method of calculating π consists of filling the area of a circle with a polygon of a greater and greater number of sides. The ratio of the area of the polygon and the square of the radius can be made arbitrarily close to the actual value of π as the number of the sides of the polygon increases (Smith, 1958). Using a 96-sided polygon, Archimedes could get the value of the ratio between the circumference and the diameter of a circle falling in between these two fractions: $\frac{223}{71} < \pi < \frac{22}{7}$ (Boyer, 1968). Working on π and on calculations for approximating the square roots of numbers, Archimedes was in fact manipulating inequalities arithmetically (Fink, 2000).

Euclid's Elements abounds in propositions that express inequality relationships between angles, sides, perimeters, or areas. However, there is no account of using inequalities in arithmetic or numbers manipulation (Fink, 2000). The contemporary translation of Euclid's words uses the inequality symbols to help the reader understand the old text, but those symbols were foreign to Euclid. The modern reader needs the symbols alongside with the text to fully see inequalities in Euclid's work. In the Pickering version of Euclid's Elements (Byrne, 1847), for example, the symbols introduced by Oughtred are used to write geometric inequalities. Proposition XXI from Euclid's Book One is chosen to exemplify some of the inequalities well known in Antiquity. Figure 1 represents one page from Byrne's (1847) The First Six Books of the Elements of Euclid. In this edition of

Euclid's works, aside from inequality symbols, Byrne uses colours to make the book attractive and appealing to students. The colours also helped the proofs that were presented as pictures. Figure 1a represents Proposition XXI from Euclid's Book One. In plain language, the proposition reads:

> If from the ends of one of the sides of a triangle, two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides (Joyce, 1996-1998).

Figure 1b is the pictorial proof of the same proposition.

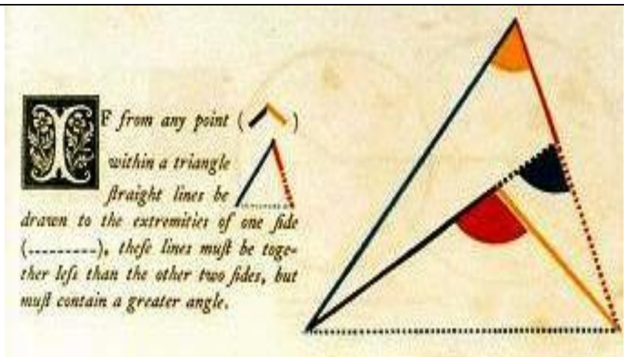
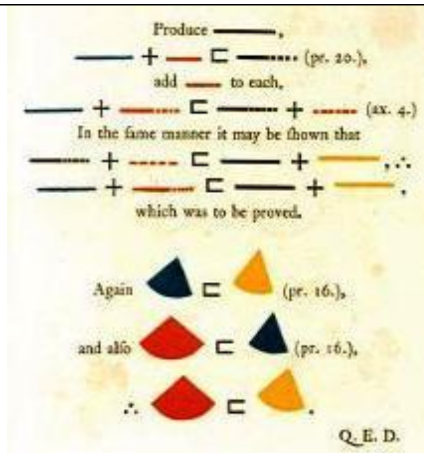
	

Figure 1a: Euclid, Proposition XXI (from Byrne, 1847, p.21)	*Figure 1b: Proposition XXI, pictorial proof*

The means inequality $\sqrt{ab} \le \frac{a+b}{2}$ as well as its proof can be seen in the next figures, 2a and 2b, respectively. The proof is based on the result that the height of a right triangle is the geometric mean of the segments that it divides the hypotenuse into. This proof of the inequality of the means looks as Euclid might have imagined (Steele, 2004). Figure 2a shows that the height of the right triangle is the geometric mean of the projections of the legs over the hypotenuse: $h = \sqrt{ab}$. From Figure 2b, the radius can be seen as the highest of the all projections from points on the circle over the diameter, thus proving the inequality $\sqrt{ab} \le \frac{a+b}{2}$.

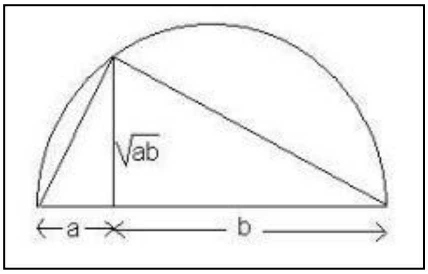


Figure 2a: *The geometric mean*

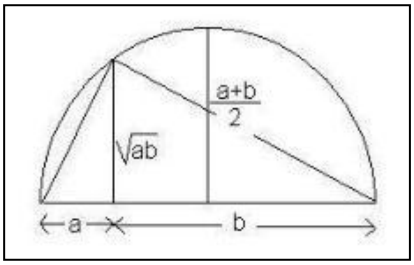


Figure 2b: The arithmetic and the geometric means

Burn's (2005) account summarizes the old history of inequalities and projects the importance of Classical work on inequalities for the further development of mathematics:

> Using inequalities to measure awkward quantities dates back to Euclid and beyond. Archimedes in particular was skilled in using inequalities to deduce equalities, and after translating his method into algebra, such proofs were used by Fermat (1636) and are accessible to undergraduates today. (p.271)

Arabic mathematicians understood the work of the Greeks and proved similar results on the volume of solids (Katz, 2009). They were also skilful in manipulating inequalities in approximations using continued fractions (Fink, 2000). Geometry, Arithmetic, and Number Theory were well established mathematics disciplines in Antiquity. However, inequalities were not recognized as sole mathematics concepts; they were only considered as peculiar tools used to develop other theories.

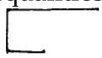
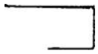
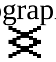
2.2 Inequalities in the Middle Years or the Development of Algebra

In the history of algebra, three developmental stages are identified: rhetorical algebra, syncopated algebra, and symbolic algebra. This division is due to Nesselmann, based on the notion of mathematical abstraction (Radford, 1997). Rhetorical algebra is the algebra of words. Syncopated algebra, the algebra of the 15^{th} and 16^{th} Centuries, uses a mixture of words and symbols to express generalities. This is the algebra of Pacioli, Cardan, and Diophantus. It is Francois Viete who introduced species and made the distinction between a constant and a variable, both being represented by letters. Viete was the first to solve parametric equations (Bagni, 2005; Sfard, 1995). Before Viete, algebra was at an operational level. As a result of Viete's contributions, equations became the objects of higher-order processes (Sfard, 1995). Viete purified algebra from all the clutter of words and presented it in abstract form, the encapsulation of a pure mathematics idea (Radford, 1997). From Viete onwards, structural algebra got its place in the history of mathematics. The structure in algebra influenced geometry. The works of Descartes and Fermat, on the shoulders of Viete, helped geometry capture generality and express operational ideas. In its early years, algebra needed geometry for reification and verification; now, geometry was using algebra for new reifications and new development (Sfard, 1995). The Middle Ages were a period of great accomplishments for algebra. A reader looking for inequalities under algebra would be surprised to discover that inequalities are missing from the picture. Algebra metamorphosed from words to symbols. Equations or identities were transformed from heavy paragraphs to delicate formulae. Inequalities, however, seem to have been left behind, forgotten, abandoned, and seemingly having no real use in the development of algebra. However, the next section shows that the development of algebra might have influenced the mathematicians to coin a symbol that have permitted inequalities to come along and evolve.

2.3 The History of the Inequality Symbol(s)

> It may be hard to believe, but for two millennia – up to the sixteenth century – mathematicians got by without a symbol for equality. They had symbols for numbers and operators – just not one for equality. (Lakoff & Núñez, 2000, p.376)

If we can imagine volumes of mathematics developed throughout the centuries without the use of the equal sign, why would it be difficult to think of inequalities being employed or produced without the use of any special symbol? This section attempts an answer to the question: When in the history of inequalities was a symbol for inequality coined, how was the symbol used and how did the symbol influence the evolution of the inequality concept?

The inequality symbols – those that are now universally accepted in mathematics literature – and they are: < for *less than*, > for *greater than*, ≤ for *less than or equal to*, ≥ for *greater than or equal to*, and ≠ for *not equal to* are already used in in this presentation. In the previous section, it was also pointed out that, at some point in the history, mathematicians using inequalities in their work adopted the symbols suggested by Oughtred, and which are:  for *greater than* and  for *less than.* It was only in the 17th Century that either one of the two types of symbols for inequality came into being. Tanner (1961) remarked that inequality, one of "the deepest-lying of the basic notions was the last to be symbolized (p.294)." "At the divide between dearth and proliferation [of inequalities] stand Harriot's inequality signs" (Tanner, 1962, p.165). In *An Introduction to the History of Mathematics,* Eves (1969) documents that the symbols < and > were first introduced in mathematics-related texts by Thomas Harriot. Harriot was a mathematician who worked for Sir Walter Raleigh as the cartographer of Virginia, North Carolina today. It is said that Harriot got inspired by the symbol  on the arm of a Native American in coining the symbols for inequalities (Johnson, 1994, p.144). The account states that Harriot decomposed the Native symbol into the two well-known symbols < and >. Tanner (1962) argues that the origin of the symbols is less mystical than that. She argues that the inequality symbols are modifications of the equal sign, a symbol which was coined by Recorde as two horizontal, parallel and equal lines, to represent that what is on one side of the sign is exactly the same as what is on the left side of it. Tanner (1962) indicates that, when producing the inequality signs, Harriot "took the equality in Recorde's sign to reside not in the two lengths, but in the unvarying distance between the two parallels" (p.166). According to Tanner (1962), Harriot modified the distance between the two lines of the equal sign, to show that the biggest quantity lies on the side of the biggest distance between the lines.

Harriot used < to represent that the first quantity is *less than* the second quantity and > to represent that the first quantity is *greater than* the second quantity (Johnson, 1994). “The symbol for ‘greater than’ is > so that $a > b$ will signify that a is greater than b. The symbol for ‘less than’ is < so that $a < b$ will signify that a is less than b” (Seltman & Goulding, 2007, p.33). Harriot was familiar with the symbolical reasoning introduced by Viete and, moreover, he transformed Viete’s algebra into a modern form (Katz, 2009). Harriot simplified Viete’s notations to the point at which even a novice in the history of mathematics could understand his formulae, whereas one needs an index of notations to understand Viete’s work. Eves (1969) considers Harriot to be the founder of “the English School of Algebraists” (p.249).

Harriot first used the symbol of inequality to transcribe the well-known inequalities of the means and then, he used the inequalities in his work to solve equations. Here are some excerpts from *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas*, Harriot’s posthumously published work.

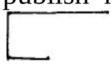
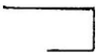
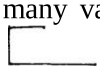
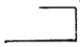
> Lemma 1
>
> If a quantity be divided into two unequal parts, the square of half the total is greater than the product of the two unequal parts.

If p and q are two unequal parts of the magnitude, then it is true that

$$\left.\begin{array}{c}\dfrac{p+q}{2}\\[4pt] \dfrac{p+q}{2}\end{array}\right| > pq$$

(Seltman & Goulding, 2007, p.96)

Transcribed, the above inequality reads: $\left(\dfrac{p+q}{2}\right)^2 > pq$.

The sample of Harriot’s work shown above may lead to a simple conclusion – that once they were coined and it was shown how they work, the inequality symbols became well established and were easily adopted. However, history shows that the mathematics community did not adopt Harriot’s symbols immediately, possibly because Harriot did not publish his work or perhaps because at the same time, in 1631, Oughtred suggested  for *greater than* and  for *less than*. Oughtred's *Clavis Mathematicae* was more popular than *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas* (The Analytical Arts Applied to Solving Algebraic Equations), Harriot’s posthumously published work (Eves, 1969). Therefore, for more than a hundred years, Oughtred's symbols were more often used than Harriot’s symbols for inequalities. Figure 2.7 b shows Oughtred's symbols at work in Pickering’s edition of Euclid’s *Elements* (Byrne, 1847, p.21). Cajori (1928-29) mentions that Oughtred's inequality symbols were hard to remember, prompting many variations of the symbols to be circulated in the literature. Oughtred himself used  for < and  for > in some parts of his work.

Many other derivations of Oughtred's symbols, as well as personal notations or improvised typewriting signs, were used to signal inequalities in the 17th and 18th centuries (Cajori, 1928-29). In the 18th century, the < and > signs finally made their way into Continental Europe (Cajori, 1928-29). Moreover, in 1734, the French geodesist Pierre Bouguer invented the symbols ≤ and ≥, to represent *less than/greater than or equal to,* respectively. These new symbols were used to "represent inequalities on the continent" (Smith, 1958, p.413). More precisely, the < symbol is used to represent quantities that are different, the first one being less than the second one. The ≤ symbol incorporates the equality as well; it allows the first magnitude to be equal to the second one.

It is well known that long before the appearance of symbolic algebra, people wrote all arguments in longhand. There were no symbols to represent the unknowns and there were no symbols to represent the relationship between unknowns as well. That was before Diophantus, during the 'Rhetorical algebra' stage (Harper, 1987). Writing mathematical statements in plain language is by no means incorrect. However, it may take several pages to describe a statement in plain language, while expressing the same statement in mathematical symbols could even take a single line. It is amazing how much the Greek mathematicians could accomplish by using rhetorical means of expressing inequalities and geometrical embodiments. Symbolic algebra produced the tools for a new embodiment of ideas and for inequalities a representation that is more abstract and specific. Moreover, the use of symbols allows for more work to be performed in a shorter time. Thus, the inequality symbol allowed for compression and aesthetic presentation of many old inequalities and spurred the development of a concept from a mere peculiarity. Radford (2006) argues that algebraic symbolism is "a metaphoric machine itself encompassed by a new general abstract form of representation and by the Renaissance technological concept of efficiency" (p.1). Efficiency helped algebra prosper, while Harriot's inequality signs stimulated the proliferation of inequalities (Tanner, 1962).

2.4 A Discipline Named Inequalities

The 18th Century is marked, in addition to a new symbol for inequalities, by the emergence of named inequalities. A new meaning is attached to the name of an inequality: the names are not a description of what the inequalities encompass, such as the 'inequality of the means'. In contrast, inequalities are assigned the names of the mathematicians who either discovered or proved them for the first time (e.g., 'Cauchy-Schwarz Inequality'). Newton's name is attached to an inequality. Cauchy's name is attached to an inequality, as well as to different proofs of other famous inequalities, due to his extensive use of inequalities in his work on limits and series (Fink, 2000). Holder, Minkowski, Hadamard, and Hardy are other mathematicians who were honoured by the use of their names to qualify different inequalities. It is important to mention that Hardy's work has been much more significant than one inequality: Hardy could be named 'the father of the Discipline of Inequalities'. He was the founder of the *Journal of the London Mathematical Society,* a proper publication for many papers on inequalities. In addition, together with Littlewood and Polya, Hardy was the editor of the volume *Inequalities,* a book that was the first monograph on inequalities. The work on the book started in 1929 and it was issued in 1934. The authors confessed that the historical and bibliographical accounts are difficult "in a subject like this, which has applications in every part of mathematics but has never been developed systematically" (Hardy, Littlewood, & Polya, 1934, p.v). Their contribution was to track down, document, solve and carefully present a volume comprising of 408 inequalities, and to officially write the first page of the history of inequalities. A question arises here: Why has the concept of inequalities been disregarded for almost two millennia before it was considered worth of special attention and developed systematically? Long before Hardy, mathematicians knew the power and importance of inequalities, since they used inequalities as tools in developing Geometry and Calculus. However, before Hardy, inequalities did not get special attention from mathematicians – nobody took the pains to introduce them to the mathematics community as a mathematical concept rather than as a simple tool used to serve other concepts. Linear or quadratic equations, for example, were studied as independent concepts by Babylonian mathematicians. Seventeen centuries before inequalities were noticed as independent mathematical concepts, Diophantus of Alexandria developed the theory of equations. It is interesting that, before Hardy, no other mathematician devoted a special interest to inequalities throughout the history of mathematics. Hardy himself attested, in his Presidential Address to the London Mathematical Society in 1928, that even though inequalities had been intensively used by analysts, there was no coherent reference to the concept.

> The elementary inequalities thus form the subject-matter of one of the first fundamental chapters in the theory of functions. But this chapter has never been properly written; the subject is one of which it is impossible to find a really scientific or coherent account. I think that it was Harald Bohr who remarked to me that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove". I will give a few examples. (Hardy, 1929, p.64)

Hardy continues by giving examples of some of the well-known and used inequalities at that time, such as the inequality of the means.

Half a century after Hardy, other tenacious mathematicians took the challenge of documenting *all* known inequalities. The product of this new collaborative work, under the supervision of Mitrinović, is a five-volume of inequalities, their proofs, evolution and applications (Fink, 2000). There would be more information to add to this short account of the history of inequalities, especially from this moment in history onwards; however, this would be more than the purpose and the size of this study.

2.5 There is more to the story

As can be summarized at this point, inequalities were first encountered in Classic Geometry, where they expressed factual relationships between quantities. The Hindu and Chinese mathematicians may have also had knowledge of those inequalities (Fink, 2000). In the big picture of the history of mathematics, Antique inequalities were captured and proved either in longhand expressions or in drawings. The Modern era is represented by the development of Algebra. To the best of my knowledge, there is no reference to any new inequality for almost two millennia, from Antiquity up to the 17^{th} Century. However, the rise of Algebra and the adoption of mathematical symbols allowed inequalities to become more easily noticed in the big picture of mathematics. With the rise of the theory of functions, inequalities seemed to have gained greater relevancy. Mathematicians began working on proving the famous Antique inequalities (e.g., Cauchy), creating extensions (e.g., Schwarz) or developing new ones (e.g., Newton, Maclaurin, & Bernoulli). Inequalities have been developed inside and through "interactions between different branches of mathematics" (Kjeldsen, 2002, p.2), like the theory of functions, linear algebra, mechanics, calculus, statistics and probability, to name only a few.

Although, there are plenty of inequalities that were produced and used over the centuries, the big production of inequalities started with the appearance of the *Journal of the London Mathematics Society*. Moreover, the first history of inequalities book was written by Hardy et al. in 1934 when edited the book *Inequalities* (Fink, 2000). Davis and Hersh (1998) argue that the production of mathematics has a rate of two hundred thousand theorems per year. For disseminating the production and proofs of inequalities, there are two journals on the topic. A library search shows 95 papers per year in one of them. It may not be too misleading to assume that more than 200 papers on inequalities are written per year, published in the last 10 years in the two inequalities journals. On top of that, there is the five-volume anthology of inequalities edited by Mitrinović, *et al* (Fink, 2000). It seems that since Hardy, the development of inequalities has been remarkable. Thus, the history of inequalities continues and the production of inequalities is ongoing.

The picture of mathematics is immense. "By multiplying the number of journals by the number of yearly issues, by the number of papers per issue and the average number of theorems per paper, their estimate came to nearly two hundred thousand theorems a year"(Davis & Hersh, 1998, p.21). This calculation takes into account only the new mathematics produced per year. Thus, taking into account all mathematics – the entire history of mathematics – makes the picture quite immense. The goal of this paper has been to capture and convey some snippets of information regarding inequalities, extracted from the grand picture of the history of mathematics. This task, although not seemingly ambitious, was not free of surprises. At first, the lens used was not pointed at the region of the picture where inequalities were expected to be. In other words, in many of the famous History of Mathematics books (Cajori, 1928-29; Katz, 2009), inequalities were not found in the index of topics. They were eventually found, however, disguised in unexpected forms. Those forms were pictures, verbal descriptions, or transcribed proofs, such as the sum of series, under Archimedes and Geometry in Katz (2009). Once the 'eye' trained enough to notice them in different eras, inequalities were seen in abundance.

The development of a mathematical concept is extremely complex and I will not claim that I have captured all relevant aspects pertaining to inequalities. However, during my research, I have located snapshots of inequalities from the big picture of mathematics and made a collage with the awareness that I was also capturing the periods of challenge in the development of or conflict within the concept. Interestingly enough, looking at the collage now, I do not perceive any problems in the evolution of inequalities. However, someone could legitimately claim that, even though there is no visible epistemological obstacle related to inequalities, the fact that it took almost two millennia for inequalities to become a discipline is itself a signal that learners might have conceptual or psychological difficulties when dealing with them (Halmaghi, 2011).

Inequalities were at first tools, and when the circumstances became favourable, they flourished into a discipline. Embedded in Geometry, they migrated to Algebra to get the power of symbols from there, and then they settled for good into the Theory of

Functions where they were enriched with new structures and philosophy. Embedded in functions, they grew omnipresent in many mathematical areas, from calculus to algebra, to statistics, to numerical analysis, to game theory. Paraphrasing Burn, I conclude this section with a historical account of the concept of inequality: *Inequality* "encapsulates methods of proofs which originated in classical Greek mathematics, developed significantly during the 17th century and reached their modern form with [Hardy]" (Burn, 2005, p.294).

FINAL REMARKS

Burn (2005) argues that not only periods of hardship, but also the actual developmental steps of a concept, can inform didactics. My search into the history of inequalities revealed no epistemological obstacles. However, the study brought out that it is recorded and documented that inequalities are not easy concepts to manipulate. Even Hardy, the man who can be called the father of inequalities, confessed:

> There are, however, plenty of inequalities which are hard to prove; Littlewood and I have had any amount of practice during the last few years, and we have found quite a number of which there seems to be no really easy proof. It has been our unvarying experience that the real crux, the real difficulty of idea, is encountered at the very beginning. (Hardy, 1929, p.64)

Thus, the answer to the investigation into the difficulty of understanding inequalities may not reside in the history of the concept, as expected. However, the answer could be deciphered from the history of inequalities in an unpredicted way. It could be the case that mathematicians are not presenting inequality to the public with the same pomp and circumstance as when presenting equation results:

> Here lies the key to the relationship between equality and inequality in mathematics, between its poetry and its prose. Mathematics is founded on inequality, the commonest thing in the world. But the kind that mathematicians most pride themselves on, finished mathematics, mathematics for show to the public, is presented as much as possible *in equality form*. (Tanner, 1962, p.164)

Inequalities are the back bones of many concepts and mathematical areas; therefore it is worth the effort of doing more research for clarifying what makes them hard to process.

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