Equations with unique solution solved using GeoGebra

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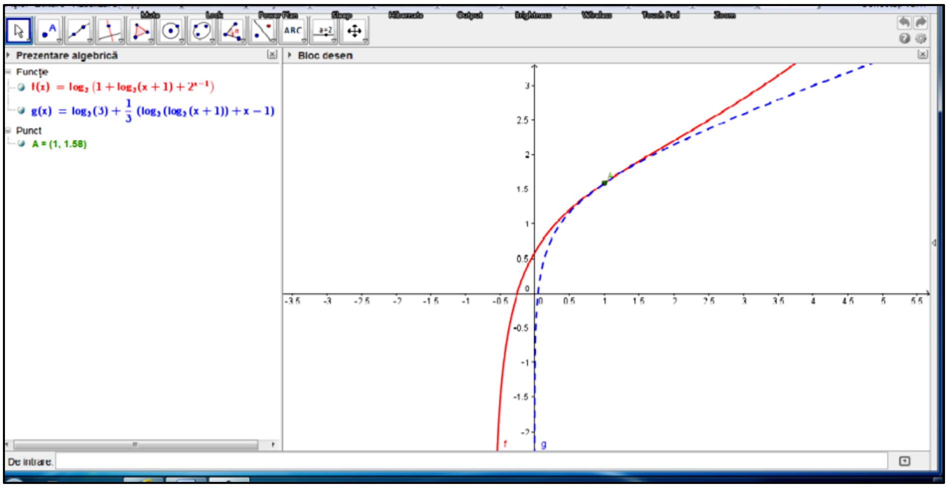
Summary: This article is a pleading for the use of GeoGebra soft in order to solve some nonstandard, difficult equations.I used it successfully to solve the exponential and logarithm equations with unique solution. It is very appropriate to use the soft at the problem analysis or investigation level, especially when the level of students is not very high.Writing the equation as an equality between two functions and analyzing their graphs, we can establish, together with the students, getting them involved, actively, in this process (by drawing graphs with the help of GeoGebra, by establishing the property of these functions based on the graphs), a strategy to solve each problem.

Statement of the first problem:

1) Solve theequation:

$$\log_2(1+\log_2(x+1)+2^{x-1}) = \log_2 3 + \frac{1}{3}(\log_2(\log_2(x+1))+x-1)$$

This equation is quite difficult, so a graphical representation of functions would be very useful to observe both solution and monotonicity of the functions.



C.E. $\begin{cases} x+1>0 \Rightarrow x>-1 \\ \log_2(x+1)>0 \Rightarrow x+1>1 \Rightarrow x>0 \end{cases}$

The equation is equivalent to

$$\log_2 \frac{1+\log_2(x+1)+2^{x-1}}{3} = \frac{1}{3}(\log_2(\log_2(x+1)) + \log_2 2^{x-1})$$
$$\Leftrightarrow \log_2 \frac{1+\log_2(x+1)+2^{x-1}}{3} = \frac{1}{3}(\log_2(\log_2(x+1)\cdot 2^{x-1})) \Leftrightarrow$$

$$\Leftrightarrow \log_2 \frac{1+\log_2(x+1)+2^{x-1}}{3} = \log_2 \sqrt[3]{1\cdot \log_2(x+1)\cdot 2^{x-1}} \quad (1)$$

But $\frac{1+\log_2(x+1)+2^{x-1}}{3} \geq \sqrt[3]{1\cdot \log_2(x+1)\cdot 2^{x-1}}$(according AM-GM inequality)

Applying $\log_2 x$ function (which is an strictly increasing function) to previous function inequality, we get:

$$\log_2 \frac{1+\log_2(x+1)+2^{x-1}}{3} \geq \log_2 \sqrt[3]{1\cdot \log_2(x+1)\cdot 2^{x-1}} \quad (2)$$

From (1) and (2) it follows that equality holds if and only if 1=$\log_2(x+1)$=2^{x-1}, so x=1, which is the unique solution.

Statement of the second problem:

2) Solve in real numbers the equation:

$$x+\log_3\left(2+\sqrt[3]{\frac{6^x}{3^x+4^x+5^x}}\right) = 5+\log_{\frac{1}{2}}\left(1+\sqrt{\frac{13^{x-1}}{5^{x-1}+12^{x-1}}}\right).$$

Solution:

Rewrite the equation as:

$$x+\log_3\left(2+\sqrt[3]{\frac{1}{\left(\frac{1}{2}\right)^x+\left(\frac{2}{3}\right)^x+\left(\frac{5}{6}\right)^x}}\right) = 5+\log_{\frac{1}{3}}\left(1+\sqrt{\frac{1}{\left(\frac{5}{13}\right)^x+\left(\frac{12}{13}\right)^x}}\right),$$

We can prove that the left hand sideis an increasing function, while the right hand side is a decreasing one.

Letx_1 and x_2 be real numbers such that$x_1 < x_2$.Using the monotonicity of the exponential function with base strictly lesst han unity we deduce that:

$$\left(\frac{1}{2}\right)^{x_1} > \left(\frac{1}{2}\right)^{x_2}$$

$$\left(\frac{2}{3}\right)^{x_1} > \left(\frac{2}{3}\right)^{x_2}$$

$$\left(\frac{5}{6}\right)^{x_1} > \left(\frac{5}{6}\right)^{x_2}$$

$$\left(\frac{1}{2}\right)^{x_1} + \left(\frac{2}{3}\right)^{x_1} + \left(\frac{5}{6}\right)^{x_1} > \left(\frac{1}{2}\right)^{x_2} + \left(\frac{2}{3}\right)^{x_2} + \left(\frac{5}{6}\right)^{x_2}$$

$$\Rightarrow \frac{1}{\left(\frac{1}{2}\right)^{x_1} + \left(\frac{2}{3}\right)^{x_1} + \left(\frac{5}{6}\right)^{x_1}} < \frac{1}{\left(\frac{1}{2}\right)^{x_2} + \left(\frac{2}{3}\right)^{x_2} + \left(\frac{5}{6}\right)^{x_2}} \Rightarrow$$

$$\Rightarrow \sqrt[3]{\frac{1}{\left(\frac{1}{2}\right)^{x_1} + \left(\frac{2}{3}\right)^{x_1} + \left(\frac{5}{6}\right)^{x_1}}} < \sqrt[3]{\frac{1}{\left(\frac{1}{2}\right)^{x_2} + \left(\frac{2}{3}\right)^{x_2} + \left(\frac{5}{6}\right)^{x_2}}} \Bigg| + 2$$

$$\Rightarrow 2 + \sqrt[3]{\frac{1}{\left(\frac{1}{2}\right)^{x_1} + \left(\frac{2}{3}\right)^{x_1} + \left(\frac{5}{6}\right)^{x_1}}} < 2 + \sqrt[3]{\frac{1}{\left(\frac{1}{2}\right)^{x_2} + \left(\frac{2}{3}\right)^{x_2} + \left(\frac{5}{6}\right)^{x_2}}} \Bigg\} \Rightarrow$$

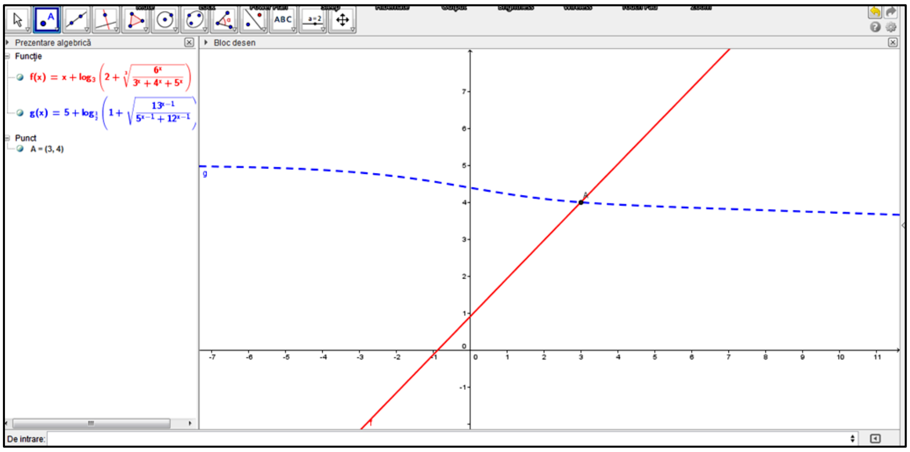
$\log_3 x$ is a strictly increasing function

$$\Rightarrow \log_3\left(2 + \sqrt[3]{\frac{1}{\left(\frac{1}{2}\right)^{x_1} + \left(\frac{2}{3}\right)^{x_1} + \left(\frac{5}{6}\right)^{x_1}}}\right) < \log_3\left(2 + \sqrt[3]{\frac{1}{\left(\frac{1}{2}\right)^{x_2} + \left(\frac{2}{3}\right)^{x_2} + \left(\frac{5}{6}\right)^{x_2}}}\right) \Bigg\} \Rightarrow$$

$x_1 < x_2$

$\Rightarrow$the function on the left hand side is a strictly increasing function.

In the same way we can prove that the function on the right hand side is strictly decreasing.



We conclude using the graphic representation that the equation has at most one solution, and it is notdifficult to guess it is x=3. It is very important to notice that the functions have different monotonicity.

For instanceif $\left.\begin{matrix} x < 3 \\ f(x) \text{ is strictly increasing} \\ g(x) \text{ is strictly decreasing} \end{matrix}\right\} \Longrightarrow \begin{cases} f(x) < f(3) = 4 \\ g(x) > g(3) = 4 \end{cases}$

Hence f(x)<g(x), ∀x< 3. We proved that the equation does not have any solutions less than 3.

Similarly we can prove that the equation does not have any solutions greater than 3.

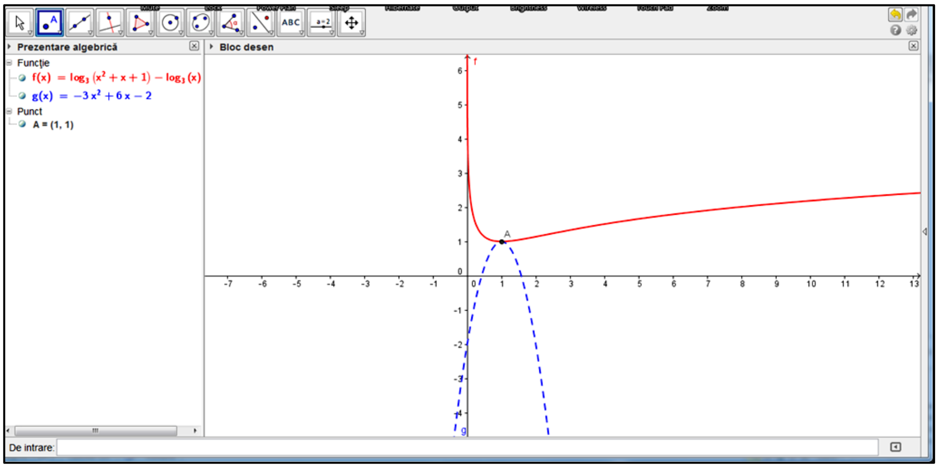
The statement of the third problem:

3) Solve the following equation
$\log_3(x^2 + x + 1) - \log_3 x = -3x^2 + 6x - 2$

Solution:
C.E. x>0

Rewritetheequation as:

$$\log_3\left(x + 1 + \frac{1}{x}\right) = -3x^2 + 6x - 2$$



Analysing the plots of the twofunctions we notice that x=1 is a unique solution of this equation and we shall formaly prove that. From the AM-GM inequality we obtain:

$$x+1+\frac{1}{x} \geq 3\sqrt[3]{x \cdot 1 \cdot \frac{1}{x}} = 3 \Longrightarrow \log_3\left(x+1+\frac{1}{x}\right) \geq \log_3 3 = 1$$

The function $-3x^2+6x-2$ is a quadratic function whose maximum is equal to $\frac{-\Delta}{4a} = 1$.

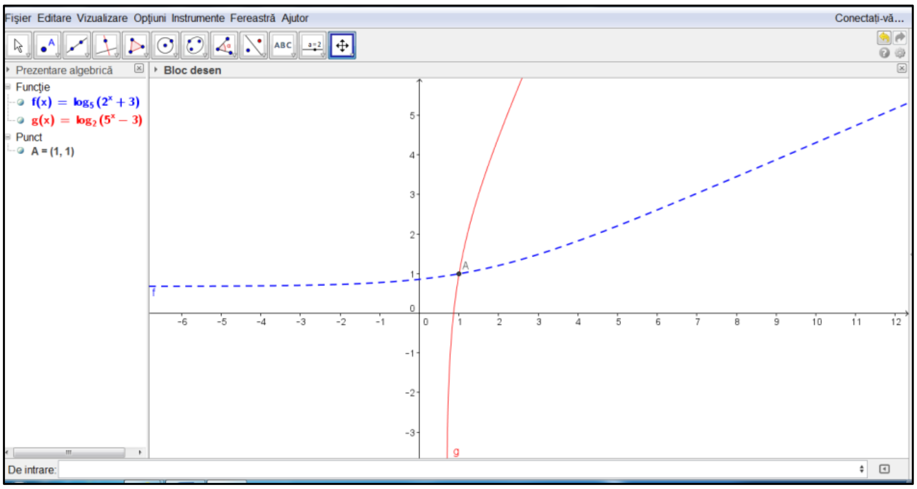
Hence the only solution is x=1.

The statement of the fourth problem:

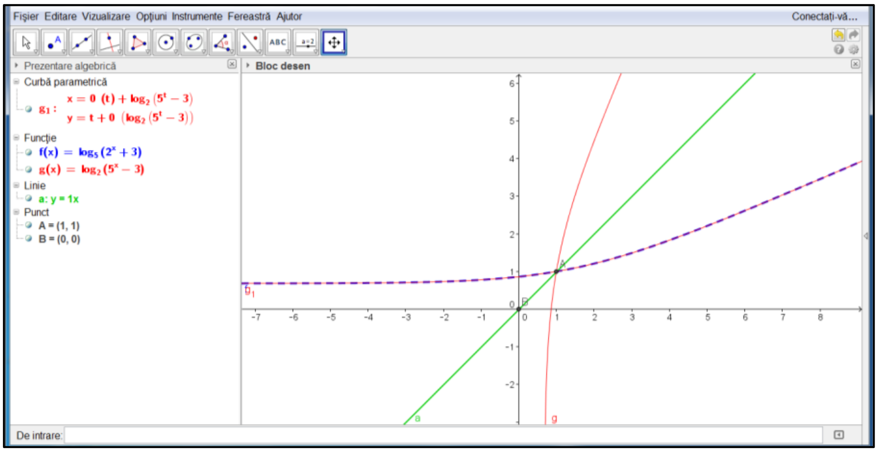
4. Solve the following equation:

$$\log_5(2^x+3) = \log_2(5^x-3)$$

Solution:



Analysing the plots of the two functions we notice that the point of intersection has x=1. On the other hand we notice that the 2 graphs are symmetrical along the first bisector. We will plot the graphs using GeoGebra, especially the 2 commands: construct a line through two points and reflect about line.



Analysing the last plot we notice that this graphs of f and g functions are symmetrical along the first bisector, hence the a functions are inverse each other.

We shall verify that $g=f^{-1}$ algebraically.

Let the function f:R→ R, f(x)=$\log_5(2^x+3)$. We shall prouve that for any y, y > $\log_5 3$ and there is an unique real x in order to have f(x)=y.

$$\log_5(2^x+3) = y \Longrightarrow 2^x+3 = 5^y \Longrightarrow 2^x = 5^y - 3 \Longrightarrow x = \log_2(5^y - 3)$$

The two functions are inverse to each other, thus the point of intersection of theirs graphs is on the first bisector. The solution of equation is equivalent to the solution of the following system of equations f(x)=g(x)=x. Observe that the obvious solutionis x=1.

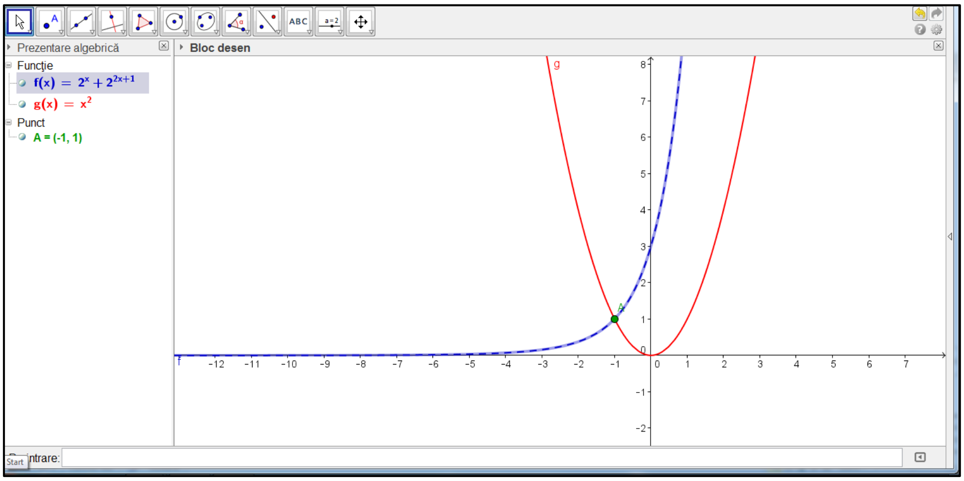
The statement of the fifth problem:

5. Solve the following equation:

$2^x + 2^{2x+1} = x^2$

Solution:

Firstly is useful to study the plots of the two functions (on the left hand sideand on the right hand side).



We can easily observe that x=-1 is the solution.

If x<-1$\Longrightarrow$ $2^x + 2^{2x+1} < 1$, while$x^2 > 1$. Thus the equation does not have any solutions less than -1.

If $x \in (-1, 0) \Longrightarrow 2^x + 2^{2x+1} > 1$ and $x^2 < 1$. Conclusion: the equation does not have any solutions in (-1, 0).

Ifx ≥ 0 ,we can study the function h(x)= $2^x + 2^{2x+1} - x^2$

h'(x)=$2^x \cdot \ln 2 + 2^{2x+1} \cdot 2 \cdot \ln 2 - 2x$

h''(x)=$2^x \cdot (\ln 2)^2 + 2^{2x+1} \cdot 4 \cdot (\ln 2)^2 - 2$

h'''(x)=$2^x \cdot (\ln 2)^3 + 2^{2x+1} \cdot 8 \cdot (\ln 2)^3 \geq 0, \forall x \geq 0$

$\Longrightarrow$ h'' is strictly increasing

$\Longrightarrow$ h''(x) $\geq 9(\ln 2)^2 - 3 > 0$ $\Longrightarrow$h'(x) is strict increasing$\Rightarrow$h'(x)$\geq$h'(0)=5 ln 2>0$\Longrightarrow$

$\Longrightarrow$h(x) is strict increasing$\Longrightarrow$h(x)$\geq$h(0)=3>0$\Longrightarrow$ $2^x + 2^{2x+1} > x^2$,

Thus the equation does not have any solutions greater or equal with 0. The only solution remains x=-1.

References:

1) Romanian Mathematical Competitions, RMC 2014
2) Ganga, Mircea, Matematica manual pentruclasa a X-a , algebra, profil M1
3) www.artofproblemsolving.com/