A COMPLEMENTARY APPROACH TO STUDY THE REAL AND IMAGINARY ROOTS OF A QUADRATIC FUNCTION

Emília de Mendonça Rosa Marques
Department of Mathematics, Univ Estadual Paulista, Bauru, **BRAZIL**
emilia@fc.unesp.br

Aguinaldo Robinson de Souza
Department of Chemistry, Univ Estadual Paulista, Bauru, **BRAZIL**
arobinso@fc.unesp.br

ABSTRACT: The influence of the linear coefficient b in the real and/or complex roots of the quadratic equation has been studied through the integrated use of GeoGebra and F(C) softwares. The F(C) software enables the visualization of real and complex roots in the Argand-Gauss plane through the use of the domain coloring technique. The graphics were obtained and analyzed simultaneously (overlapped) in order to complement the information provided on each approach with the mapping in R → R and in C → C with GeoGebra and F(C), respectively. This methodology allows the simultaneous study of real and complex roots of the quadratic function with the possibility of observing the behavior change of these roots with the variation of the function coefficients.

KEYWORDS: Argand-Gauss plane, GeoGebra, Complex Function.

1. Introduction

The importance of the study of functions can be stated in Mathematics as, for example, in Real and Complex Analysis and Numerical Calculation, and in other areas of Science such as Physics, Chemistry and Biology. As one example, in Physics, we can mention the study of the simple pendulum behavior [1], in which there is an attempt to understand the variation of the oscillation amplitude, θ, as a function of the oscillation period, T, in the time unit t (Equation 01).

$$\Theta_t = \Theta_0 \ \cos(2\pi t/T) \qquad (01)$$

In which Θ_t, and Θ_0 are the oscillation amplitudes in time t, and zero (0), respectively.

One important example in Chemistry is the study of the energy and shape of atomic and molecular orbitals described by the wave function, ψ, proposed by Erwin Schrödinger [2], which describes the energy E of an orbital when applying the Hamiltonian Operator H (Equation 02).

$$H\,\psi = \mathrm{E}\,\psi \qquad (02)$$

In the present work, we study the quadratic function, $ax^2 + bx + c$, with emphasis on the behavior of the real and complex roots when varying the linear coefficient b in a complementary approach, by using the GeoGebra and F(C) software mapping in the $R \rightarrow R$ and $C \rightarrow C$, respectively. The present work is divided into the following sections: Grounds, Quadratic Function, GeoGebra and F(C) softwares, Methodology, Results; Conclusions, Possible Future Work, and References.

2. Grounds

2.1 The roots of quadratic functions

The quadratic function is a polynomial function of the type $f(x) = ax^2 + bx + c$, with $a \neq 0$ and, according to the fundamental theorem of algebra, it has two solutions which can be either real or complex and can be obtained from the quadratic formula. In Figure 1 we present the graph of the parabola C: $x^2 - 2y = 1$, with some of its features: the axis of symmetry (OY), the point O of the intersection with its axis termed the vertex of the parabola, the axis of the parabola (B), and the focus (A) [3].
The polynomial function associated with this parabola is given by:
$f(x) = 0.5x^2 + 0.5$.

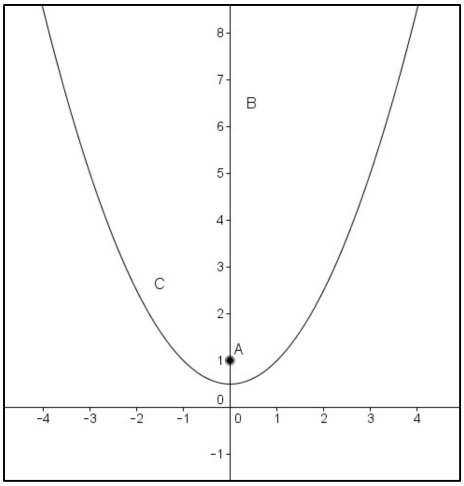

Figure 1. Graph of the parabola C: $x^2 - 2y = -1$, with its features.

The solution of the unreduced quadratic equation (in which $a \neq 1$) can be obtained by using the Baskara formula to find the two roots. By using the Baskara formula we can determine the solution of a quadratic equation, in which we can observe three possible solutions: two real and distinct roots, two equal real roots, or a pair of complex conjugated roots. [3].

The quadratic equation can be solved by many methods including square completion, discriminant calculation, quadratic factorization, graphical inspection for real roots, trigonometric and geometric solution, and solution with continued fractions [4].

2.2 The GeoGebra software

The GeoGebra software, developed by Markus Hohenwarter and collaborators, is a dynamic mathematics software that allows us to study concepts in Geometry, Algebra and Calculus simultaneously through a graphical interface. This interface provides three different windows for operating with mathematical objects: the graphical, the algebraic and the spreadsheet areas, that allow representation of dots, graphs of functions, coordinates of dots, equations, etc. [5]. In Figure 1 we represent a display of the interface with the three application zones describing the function $y = x^2 + 3x + 1$ and its corresponding graph.

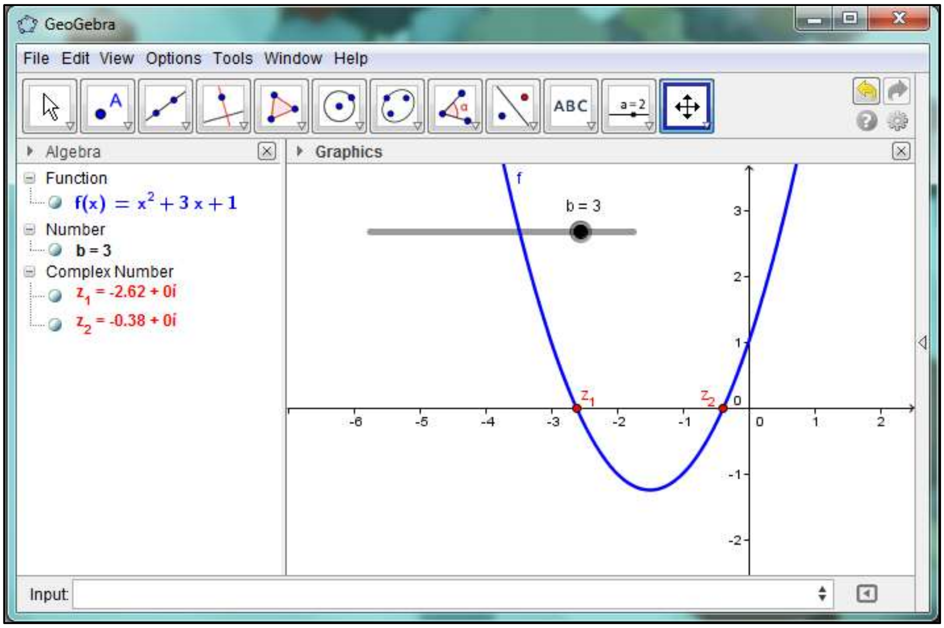

Figure 1. *GeoGebra* software interface showing the graph of the real function $y = x^2 + 3x + 1$.

2.3 The F(C) software

This software, developed by the authors of this work, allows the study of functions with a complex variable by displaying the behavior of notable dots such as the real and/or complex roots in the Argand-Gauss plane using the domain coloring technique [6, 7]. It was called the Complex Plan Colors Map, which is related to the chosen color palette. This is a biunivocal colors distribution of dots on the plane, in which each complex number is identified by a unique characteristic color, and the different shades of the same color represent different complex numbers [8]. We illustrated the Colors Map in the Argand-Gauss plane in Figure 2 below representing the *f(z) = z* function.

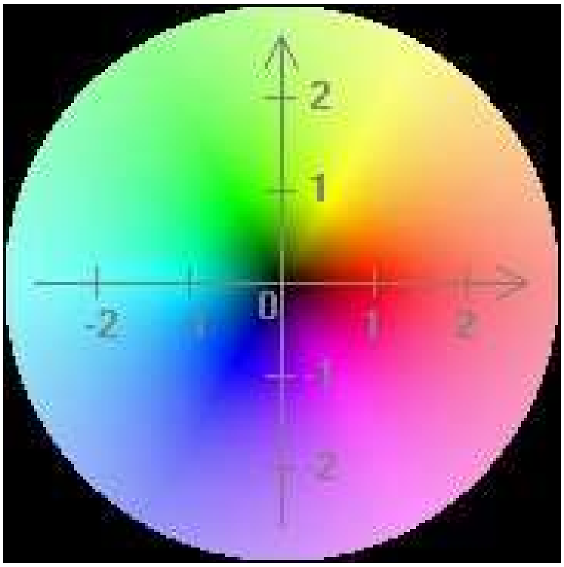

Figure 2. Color map of the Argand-Gauss plane for *f(z) = z*.

The Colors Map reading proceeds as follows: when referring to the complex number 1 + 0i, corresponding to the position (1.0), we use the red color in its primary hue. The other positive real numbers are related to the red color and its tone becomes lighter when your module increases and darker when it tends to zero. For the complex number 0 + 0i, that is, the position (0.0), this is represented by the color black (the darkest shade of all colors in the map) and so on for the other colors and their various shades. This type of association is quite useful when representing the function to be studied, mainly due to the dimensional aspect occupied by the color [9].

Consider the function *f*: C → C (complex of a complex variable) given by *f(z) = w*. The variables *z* and *w* are complex ones and, in order to view the graph of this function, we need two dimensions for each variable, which makes this representation unfeasible. So the proposed model assumes that the variable *z* must be represented by the coordinates *x* and *y* and the variable *w* by a color and hue obtained in the color map (Figure 2).

As an example, considering the function *f (z) = - z*, Table 1 below shows some features of this function.

Table 1. The domain for f (z) = - z.

Point	Position on the plane (Domain) *z*	Position on the Map (Image) *f(z)*	Color of the image printed in the domain
A	*1 + i*	*-1 – i*	*Blue*
B	*2 + i*	*-2 - i*	*Cyan-Blue*
C	*0*	*0i*	*Black*
D	*-1 - 2i*	*1 + 2i*	*Greenish yellow*

The calculation of the colored domain of the function *f (z) = -z* was done with this technique by using the values of the variables *z* and *w* as position and color, respectively, in the Argand-Gauss plane.

In Figure 3 bellow, we can see the colored domain for the function *f (z) = -z* along with the points A, B, C and D.

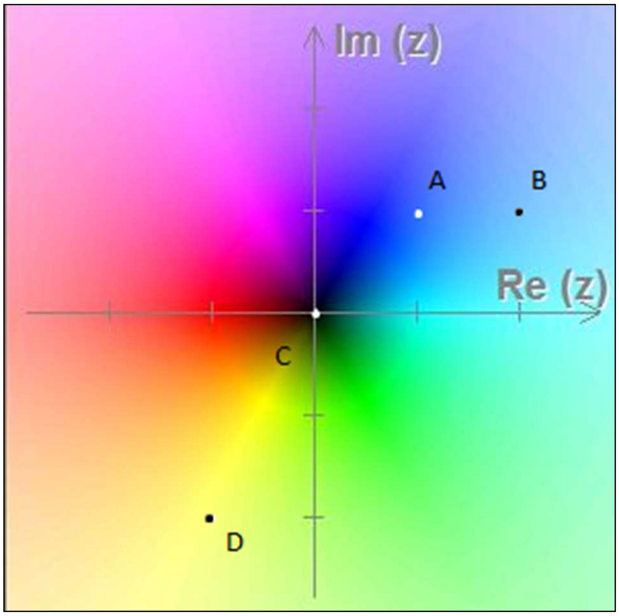

Figure 3. The function *f(z) = -z* in the colored domain of the Argand-Gauss plane.

This representation method of complex functions can be used in the process of teaching and learning Mathematics in the field of Real and Complex Analysis and in the study of physical phenomena such as fluid flow [10] and quantum mechanics [11].

3. Results and Discussion

Each of the colored domains here presented were generated using the F(C) software and varying the linear parameter, *b*, in the interval *-3* to *3,* recording an image for each of the parameters at one-unit intervals. With the support of *GeoGebra*, an Applet was created for the quadratic function using a selector for the linear parameter b. For each different value of the parameter *b*, an image was generated in each software, *GeoGebra* and *F(C)*. Later, the images were overlapped respecting the scales.
Figures 4a to 4g represent the results obtained by the overlapping of the colored domains of the complex quadratic functions, by using the *F(C)* software, and the graphs of the same function limited to the real numbers obtained, built with *GeoGebra*. In these figures, we considered the variation of the linear parameter, $-3 < b < 3$, at one-unit intervals.

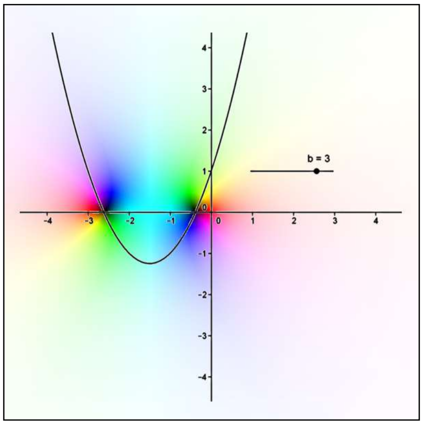

Figure 4a. $f\left(z\right) = z^2 + bz + 1$; *b=3*

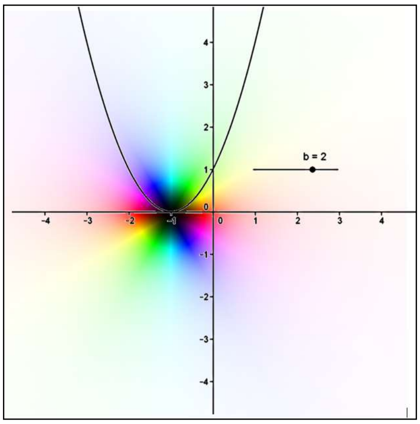

Figure 4b. $f\left(z\right)=z^2+bz+1$; *b=2*

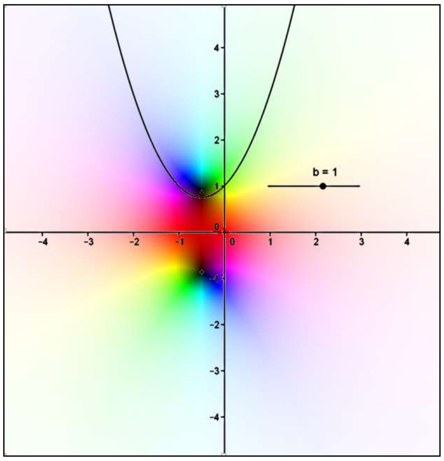

Figure 4c. $f\left(z\right)=z^2+bz+1$; *b=1*

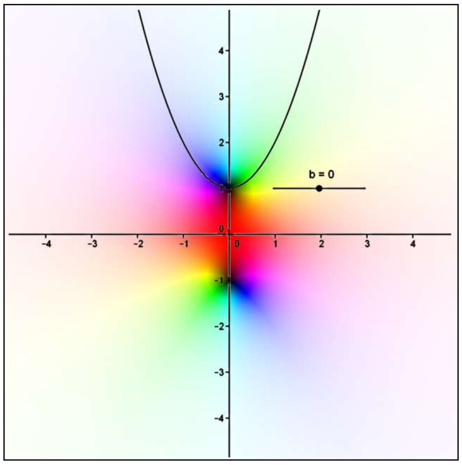

Figure 4d. $f\left(z\right)=z^2+bz+1$; *b=0*

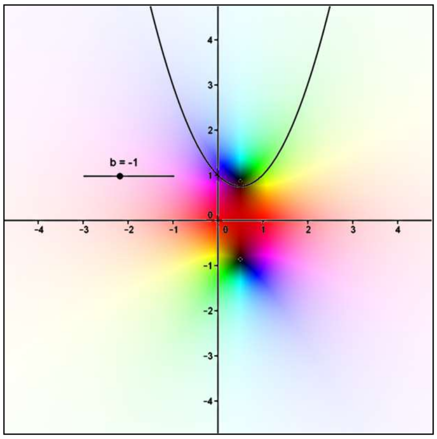

Figure 4e. $f\left(z\right)=z^2+bz+1$; *b=-1*

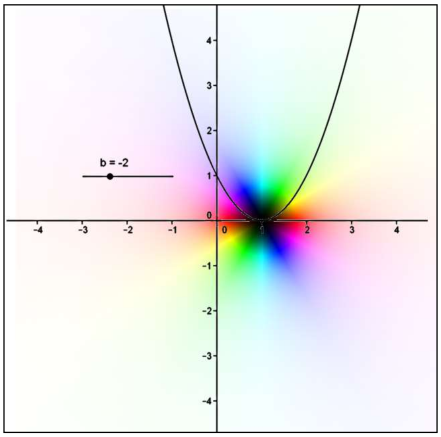

Figure 4f. $f\left(z\right)=z^2+bz+1$; *b=-2*

By looking at Figures 4a and 4g, it is possible to observe that the interval of the x-axis between the real roots of the quadratic function always assumes a real negative value (cyan color), as stated in the theory, since the values of the real roots are real values of the opposite sign to the quadratic coefficient of the function. Now, it is also shown through the colors, which is a highly efficient way to emphasize this property. By analyzing Figures 4c, 4d and 4e, it becomes evident that, for complex numbers on the tracking line that connects the complex conjugated roots, the values given by the quadratic function are always real and positive (red color). This fact is shown algebraically (Equation 06).

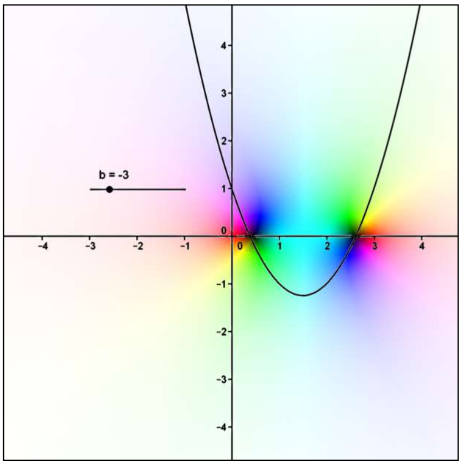

Figure 4g. $f(z)=z^2+bz+1$; *b=-3*

It was also observed that being $z_1=(a_1,b_1)\in C$ one of the complex roots of the quadratic equation $ax^2+bx+c=0$, the complex conjugate number $z_2=\overline{z_1}=(a_1,-b_1)\in C$ has another complex root. That is, the image of the function is zero for both z_1 and z_2, whence derives the equation 06.

$$\left.\begin{aligned} f(a_1+b_1i) &= (a_1+b_1i)^2+b(a_1+b_1i)+c= \\ &= a_1^2-b_1^2+2a_1b_1i+a_1b+bb_1i+c \\ &= (a_1^2-b_1^2+a_1b+c)+(2a_1b_1-bb_1)i=0 \end{aligned}\right\} \Rightarrow \begin{cases} a_1^2-b_1^2+a_1b+c=0 \\ 2a_1b_1-bb_1=0 \end{cases}$$

(03)

Thus, it can be said that the sentence in (03) is true.

$$2a_1b_1-bb_1=0 \Leftrightarrow q(2a_1k-bk)=0 \Leftrightarrow 2a_1k-bk=0 \text{ , for } k=\frac{b_1}{q},\ q\neq 0$$

(04)

As the complex numbers on the tracking line that joins such complex conjugate roots are from the type $z=(a_1,k)$, being $|k|\leq b_1$, it can be stated, considering (04), that the image of these complex numbers is always a pure real number (equation 05).

$$
\begin{aligned}
f\left(a_1+ki\right)=\left(a_1+ki\right)^2+b\left(a_1+ki\right)+c= \\
=a_1^2-k^2+2a_1ki+a_1b+bki+c= \\
=\left(a_1^2-k^2+a_1b+c\right)+\left(2a_1k-bk\right)i \\
=\left(a_1^2-k^2+a_1b+c\right)\in R
\end{aligned}
\tag{05}
$$

We still have $\left|k\right|\leq b_1$, then

$$
\left(a_1^2-k^2+a_1b+c\right)\geq\left(a_1^2-b_1^2+a_1b+c\right)=0 \tag{06}
$$

We can observe that the roots of a quadratic function can be obtained by the known formula of Baskara:

$$
z=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad , \tag{07}
$$

It presents the two complex conjugate roots, when $b^2<4ac$, the two distinct real roots, when $b^2>4ac$ and multiple real roots when $b^2=4ac$.
By doing the identification of complex numbers with the Cartesian plane, the following representation for the complex conjugate roots is obtained:

$$
z_1=\left(\frac{-b}{2a},\frac{\sqrt{4ac-b^2}}{2a}\right) \quad \text{and} \quad z_2=\left(\frac{-b}{2a},\frac{-\sqrt{4ac-b^2}}{2a}\right) \quad ,
$$

$$
\text{for } b^2<4ac \tag{08}
$$

So, squaring each term, both real and imaginary, of these complex numbers (equation 08) and adding them, it has to be:

$$
\left(\frac{-b}{2a}\right)^2+\left(\frac{\pm\sqrt{4ac-b^2}}{2a}\right)^2=\frac{b^2+4ac-b^2}{4a^2}=\frac{c}{a}=r \tag{09}
$$

Therefore, it can be concluded that these roots define a circle centered at the origin and radius $r={}^{c}\!/_{a}$ as the geometric locus in the plane, according to the variation of the linear parameter.
Using *GeoGebra*, the results obtained algebraically in equation (09) can be dynamically displayed. Figure 5 illustrates the complex roots ranging over a circle of radius 1, considering the quadratic function ax^2+bx+c with *a* and *c* equal to 1.

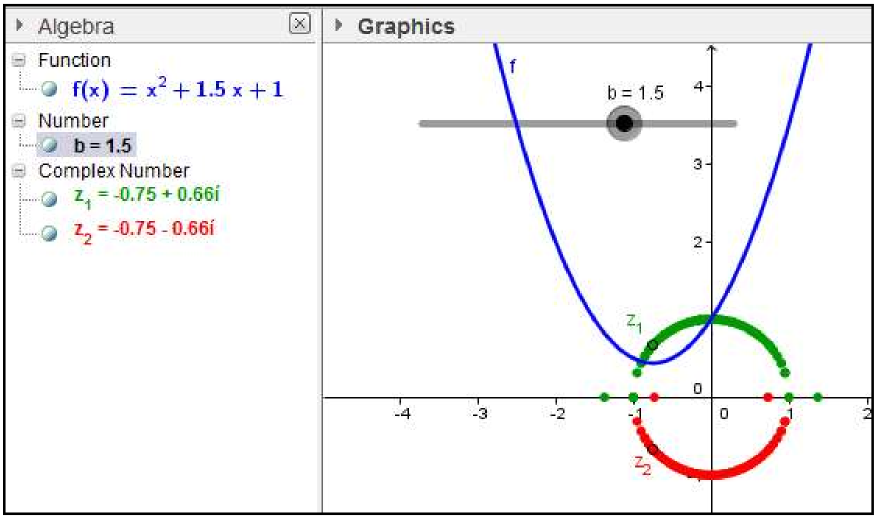

Figure 5. Graphical representation of the complex roots; -2 < b < 2.

4. Conclusions

Through the colors obtained around the roots of the function, whether they are real or complex conjugates, we can observe the diametrically opposite symmetry whose center is the midpoint between the real or complex roots. Finally, was also observed that the variation of the linear parameter used for quadratic functions promotes the movement of the complex roots making them describe a circle of radius 1, considering $a = c = 1$. Therefore, the combined use of *GeoGebra* and *F(C)* softwares enables a more comprehensive understanding of algebraic properties involved in physical and chemical phenomena that are modeled by polynomial functions. It is also evident that the algebraic work will be more meaningful since, based on the geometric approach (visual), it is possible to make inferences, thus elucidating the questions "why?", "for what?" and "how?" that should be asked to algebraically prove such inferences. This new methodology enables another way of teaching and learning real and complex functions and can be used in the classroom for the development of mathematical concepts involving the calculation of real and imaginary roots of quadratic equations.

5. References

[SRS12] Santos, R. C. M.; de Souza, A.R. (2012). Estudo do pêndulo simples com auxílio do software GeoGebra na abordagem dos Estilos de Aprendizagem. In: Actas de La Conferência Latinoamericana de GeoGebra. Montevidéu: Uruguai.

[McQ83] McQuarrie, D. A. (1983). *Quantum Chemistry*. Mill Valley, California: University Science Books.

[Vyg87] Vygodsky, M. (1987). *Mathematical Handbook*. Moscou: Mir Publishers.

[Wall48] Wall, H. S. (1948). *Analytic Theory of Continued Fractions*, D. Van Nostrand Company, Inc.

[Hoh13] Hohenwarter, M. (2013), Marktanalysen Fur Open Source Software Im Osterreichischen Gesundheitswesen, GRIN Verlag.

[SSM08] da Silva, E. L.; de Souza, A. R.; Marques, E. M. R. (2008). *Números e Funções Complexas: Representação e Interpretação Gráfica*. São Paulo: Cultura Acadêmica Editora.

[SSM09] da Silva, E. L.; de Souza, A. R.; Marques, E. M. R. (2009). Alguns estudos de fluxo de fluídos utilizando software gráfico. *Revista Brasileira de Ensino de Física, v. 31, n. 3*, 3502-3509.

[Nee02] Needham, T. (2002). *Visual Complex Analysis*. Oxford: Oxford University Press.

[SSR06] da Silva, E. L.; de Souza, A. R. (2006). *Funções de uma variável complexa: visualização e interpretação gráfica.* Recovered (2013), in (wwwp.fc.unesp.br/~edvaldo/index.htm).

[SFZ09] Smith, Richard.L., Fang, Zhen. (2009). Techniques, applications and future prospects of diamond anvil cells for studying supercritical water systems. *Journal of Supercritical Fluids*, v.47, ed 3, pp 431-446.

[Tha05] Thaller, B. (2005). *Advanced Visual Quantum Mechanics*. New York: Springer.