Assess high-order thinking development of students in teaching mathematics at high schools in Vietnam

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Abstract. In teaching Mathematics at high school in Vietnam, beside from pedagogical skills, teachers also need assessment competence for student's higher-order thinking. Assessing developing student's thinking is a part of teaching and learning process, it play a role as feedback. It also has an active role in regulating program content, teaching and learning methods towards the thinking advancement of students. This paper proposed some examples of applying mathematical knowledge in assessing developing student's thinking.

Keywords: *Assessment competence, higher-order thinking, mathematics teaching methods.*

1. Introduction

For the difficulties in evaluating higher-order thinking in students, a wide number of researchers have used experimental case studies as a basis for studying assessment. Tests, grades, or the answers to specific questions are typically used as instruments in their studies. For instance, (Newmann et al., 2001) used a statewide basic skills test given to third, sixth, and eighth grade students, to delineate between intellectual and non-intellectual questioning. They categorized the testing assessment process into two types – didactic, where students are evaluated based on their ability to recall definitions, know rules, or state facts, and interactive, in which problem solving and reasoning ability are required. A similar study was performed at the fourth-grade level by Wenglinsky (2004) – he examined the correlation between instruction emphasizing higher-order thinking and student performance on large-scale measures. Using the National Assessment of Educational Progress and the Trends in International Mathematics and Science Study as his evidence, he suggested that there was a relationship between emphasis on thinking approaches and student performance.

Some research studies have found that faculty members are the primary reason for difficulties in assessing higher-order thinking. A study performed by (Thompson, 2008) examined the interpretation of Bloom's taxonomy by mathematics teachers in the southeastern U.S. The author found that math teachers do not fully understand the meaning of higher-order thinking and thus, have difficulty in creating test items for

students.
Finally, there have been a number of researchers examining computer tools for assessing higher-order thinking in students. (Thomas et al., 2002) incorporated a Java-based online assessment tool in their teaching of college-level statistics. (Forgasz, Prince, 2002) investigated a number of different software tools for assessing the understanding of mathematics. (Lee et al., 2004) performed a similar study by integrating educational video games into second- grade mathematics classes to assess thinking skills. (Rice, 2007) suggests that the virtual appeal of computerize gaming environments can be used to improve cognitive skills. In addition, he identified several popular mathematics games in the computer market that failed to make this step. Others, such as (Collis, 1998; Yazdani, 1999; Oliver, McLoughlin, 2000), also promoted online virtual computer programs as tools for learning higher-level mathematical
As teachers we can support our students to develop the higher-order thinking skills they need to tackle problems by the classroom math we create. Many writers have attempted to clarify what is meant by a higher-order thinking skills approach to learning mathematics. The focus is on teaching mathematical topics through problem-solving to develop higher-order thinking skills for contexts which are characterised by the teacher 'helping students construct a deep understanding of mathematical ideas and processes by engaging them in doing mathematics: creating, conjecturing, exploring, testing, and verifying. (Dewey, 1933) state that, complex real-life problems often demand complex solutions, which are obtained through higher level thinking processes. Teaching higher order thinking, then, provides students with relevant life skills and offers them an added benefit of helping them improve their content knowledge, lower order thinking, and self-esteem (DeVries, Kohlberg, 1987; McDavitt, 1993; Son, VanSickle, 1993).
(King et al., 1998) higher-order thinking includes critical, logical, reflective, metacognitive, and creative thinking. (It is) activated when individuals encounter unfamiliar problems, uncertainties, questions, or dilemmas; (Lopez, Whittington, 2001), higher-order thinking occurs when a person takes new information and information stored in memory and interrelates and/or rearranges and extends this information to achieve a purpose or find possible answers in perplexing situations; (Thomas, A., Thorne, G., 2000) …higher-order thinking takes thinking to higher levels than just restating the facts. (It) requires that we do something with the facts. We must understand them, connect them to each other, categorize them, manipulate them, put them together in new or novel ways, and apply them as we seek new solutions to new problems"; (Kruger, K., 2013) higher-order thinking involves "concept formation, critical thinking, creativity/brainstorming, problem solving, mental representation, rule use, reasoning, and logical thinking.
We state that, higher-order thinking takes thinking to higher levels than restating the knowledge and requires students to do something with the knowledge - understand them, infer from them, connect them to other the knowledge and concepts, categorize them, manipulate them, put them together in new ways and apply them as we seek new solutions to new problems or new perplexing situations.

2. Research contents and discussion

Student as a mathematics teacher candidate must have competence in solving and creating mathematical problems. On the one competence, problem solving ability is reviewed by using four aspects: (1) conceptual and procedural understanding, (2) strategy knowledge, (3) mathematical communication ability and (4) accuracy. On the other competence, student's creativity in developing mathematical problem is evaluated based on: (1) fluency, (2) flexibility and (3) novelty (Sugiman S., 2013).
When you teach and assess higher-order thinking regularly, over time you should see benefi ts to your students. Your understanding of how your students are thinking and processing what they are learning should improve as you use assessments specifi cally designed to show students' thinking. Ultimately, their thinking skills should improve, and so should their overall performance. Students learn by constructing meaning, incorporating new content into their existing mental representations; therefore, improving thinking skills should actually improve content knowledge and understanding as well.

2.1. General principles for assessing higher-order thinking

Constructing an assessment always involves these basic principles:
- Specify clearly and exactly what it is you want to assess.
- Design tasks or test items that require students to demonstrate this knowledge or skill.
- Decide what you will take as evidence of the degree to which students have shown this knowledge or skill.
- This general three-part process applies to all assessment, including assessment of higher-order thinking. Assessing higher-order thinking almost always involves three additional principles:
    - Present something for students to think about, usually in the form of introductory text, visuals, scenarios, resource material, or problems of some sort.
    - Use new knowledge - knowledge that is new to the student, not covered in class and thus subject to recall.
    - Distinguish between level of difficulty (easy versus hard) and level of thinking (lower-order thinking or recall versus higher-order thinking), and control for each separately.

2.2. Examples of activities that promote higher-order thinking

With innovative approaches to teaching high school, student-centered and create excitement in learning. Students actively dominate knowledge. Therefore, teaching the student grasp the essence of a mathematical concept is very important (N.H. Hau, et al., 2015).
We give out some examples by (Nhi D.V.,Tinh T.T., et al., 2015) that help pupils develope their higher-order Thinking while teaching some topics of mathematics in schools in Vietnam.

- *Example about using convex function and Jensen's inequality*

(Yuly, 2010) state that, remembering is the lowest level of learning in the cognitive domain in Bloom's Taxonomy and typically does not bring about a change in behavior. It involves memorization and recall of information with no evidence of understanding. Students absorb, remember, recognize and recall information. However, it is the building block of all subsequent levels of learning because the students must remember information presented before progressing to the next levels.

Definition 1. Function $y = f(x)$ call is a *convex function* in $(a;b)$ if forall $a < x_1, x_2 < b$ and forall $\alpha \in (0;1)$.

We have

$$\alpha f(x_1) + (1-\alpha) f(x_2) \geq f(\alpha x_1 + (1-\alpha)x_2).$$

Definition 2. Function $y = f(x)$ call is a *concave function*, in $(a;b)$ if forall $a < x_1, x_2 < b$ and forall $\alpha \in (0;1)$ we have

$$\alpha f(x_1) + (1-\alpha) f(x_2) \leq f(\alpha x_1 + (1-\alpha)x_2).$$

Students comprehend the meaning of the mathematical knowledge presented and predict consequences or effects from it. No change in behavior occurs at this level. Students are able to describe their understanding of what is presented and discuss how the new knowledge learned may or may not work in their own environment. This type of thinking skills tells you that a student can grasp and interpret prior learning.

Proposition 1. Assume, $y = f(x)$ defined and continuous in $(a;b)$ where $a < b$. Function $y = f(x)$ is a convex function in $(a;b)$ if and only if

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

or $\begin{vmatrix} 1 & x_1 & f(x_1) \\ 1 & x & f(x) \\ 1 & x_2 & f(x_2) \end{vmatrix} \geq 0$ forall $x_1, x, x_2 \in (a;b)$ such that $x_1 < x < x_2$.

Proof:. Assume $y = f(x)$ is convex function in $(a;b)$. Where $x_1, x, x_2 \in (a;b), x_1 < x < x_2$, we have

$$x=\frac{x_2-x}{x_2-x_1}x_1+\frac{x-x_1}{x_2-x_1}x_2, f(x)\le\frac{x_2-x}{x_2-x_1}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2).$$

Deduce we get inequality

$$(x_2-x)f(x_1)+(x_1-x_2)f(x)+(x-x_1)f(x_2)\ge 0$$

or namely $\begin{vmatrix}1 & x_1 & f(x_1)\\ 1 & x & f(x)\\ 1 & x_2 & f(x_2)\end{vmatrix}\ge 0.$ The opposite is always right.

Proposition 2. Assume $y=f(x)$ defined and continuous in $(a;b)$ and it has a finite derivative $f'(x)$. We have $y=f(x)$ is convex function if and only if $f'(x)$ is non-decreasing function in $(a;b)$.

Proof: Assume $y=f(x)$ is a convex function in $(a;b)$.

Where $x_1,x,x_2\in(a;b), x_1<x<x_2$, we have $x=\frac{x_2-x}{x_2-x_1}x_1+\frac{x-x_1}{x_2-x_1}x_2$

and $\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}$.

We deduce

$$f'(x_1)=\lim_{x\to x_1}\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\lim_{x\to x_2}\frac{f(x_2)-f(x)}{x_2-x_2}=f'(x_2).$$

We deduce $f'(x_1)\le f'(x_2)$. Contradicts the assumption $f'(x)$ is non-decreasing function in $(a;b)$. With $x_1,x,x_2\in(a;b), x_1<x<x_2$ we have $\frac{f(x)-f(x_1)}{x-x_1}=f'(\alpha)$ and $\frac{f(x_2)-f(x)}{x_2-x}=f'(\beta)$, where $x_1<\alpha<x<\beta<x_2$.

Because $f'(\alpha)\le f'(\beta)$ deduce $\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}$.

From proposition 1, we deduce $y=f(x)$ is convex function.

Remembering and understanding often go together, but Understanding goes one step beyond remembering. Students effectively apply concepts, principles, methods, rules, laws, theories, and other newly learned information to novel and concrete situations

in the form of measurable activity with minimal direction. In this stage, a change in behavior occur and it require a higher level of understanding than those in the mathematics knowledge and understand domains.
From proposition 2 we deduce the following results:

Theorem 1. Supposed $y = f(x)$ defined and continuous in an interval I. Assume $f(x)$ has a continuous derivative $f'(x)$ and $f''(x)$ is finite in an interval I. We have $y = f(x)$ is a convex function if and only if $f''(x) \geq 0$ in I.

Theorem 2. [Jensen] If $y = f(x)$ is a convex function in $(a;b)$ then forall $a_1, \ldots, a_n \in (a;b)$ and forall real $\alpha_1, \ldots, \alpha_n \geq 0, \sum_{k=1}^{n} \alpha_k = 1, n \geq 2,$ we have

$$\alpha_1 f(a_1) + \alpha_2 f(a_2) + \cdots + \alpha_n f(a_n) \geq f(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n).$$

Proof: Using inductive method according to n. When $n = 2$ then it is right. Assume it is right by $n \geq 2$. we consider $n+1$ points $a_1, \ldots, a_n, a_{n+1} \in (a;b)$ and some reals $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \geq 0, \sum_{k=1}^{n+1} \alpha_k = 1$ and $\alpha_{n+1} > 0$. Set

$$b_n = \frac{\alpha_n}{\alpha_n + \alpha_{n+1}} a_n + \frac{\alpha_{n+1}}{\alpha_n + \alpha_{n+1}} a_{n+1} \in (a;b).$$

From supposed of inductive, we have

$$\begin{aligned} & f(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{n-1} a_{n-1} + \alpha_n a_n + \alpha_{n+1} a_{n+1}) \\ & = f(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{n-1} a_{n-1} + (\alpha_n + \alpha_{n+1}) b_n) \\ & \geq \alpha_1 f(a_1) + \alpha_2 f(a_2) + \cdots + \alpha_{n-1} f(a_{n-1}) + (\alpha_n + \alpha_{n+1}) f(b_n). \end{aligned}$$

Because $f(b_n) = f(\frac{\alpha_n}{\alpha_n + \alpha_{n+1}} a_n + \frac{\alpha_{n+1}}{\alpha_n + \alpha_{n+1}} a_{n+1}) \geq \frac{\alpha_n f(a_n)}{\alpha_n + \alpha_{n+1}} + \frac{\alpha_{n+1} f(a_{n+1})}{\alpha_n + \alpha_{n+1}}$

deduce $\sum_{k=1}^{n+1} \alpha_k f(a_k) \geq \sum_{k=1}^{n+1} f(\alpha_k a_k)$. Hence, ends the proof.

Applying entails using what has been learned in new situations. Asking students to consider a newly learned fact as they build or make something can foster this level of thinking.
Analyzing involves thinking about a whole in terms of its various parts. You can

encourage this level of thinking by asking students what mathematics knowledge could be used for some mathematical problems.

Remark 1. If it is a concave function then we have reverse inequality.

Students understand the use of variables in mathematical expressions. Students analyze situations and they to describe relationships. They write expressions and inequalities that correspond to given situations, evaluate expressions, and use expressions and formulas to solve problems. Students understand that expressions in different forms can be equivalent, and they use the properties of operations to rewrite expressions in equivalent forms. Students use properties of operations and the idea of maintaining the inequalities of both sides of a inequalities to solve simple one-step inequalities.

Example 1. *Prove that, if $a,b,c \geq 1$ then we have*

$$T = \sqrt[3]{\frac{1}{(1+a)^8}} + \sqrt[3]{\frac{1}{(1+b)^8}} + \sqrt[3]{\frac{1}{(1+c)^8}} \geq 3\sqrt[3]{\frac{1}{(1+\sqrt{abc})^8}}.$$

Proof:. Because $f(x) = \sqrt[3]{x^8}$ is a monotonically increasing function and convex.

With $a,b,c \geq 1$ we have

$$T \geq 3\sqrt[3]{\left[\frac{\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}}{3}\right]^8} \geq 3\sqrt[3]{\frac{1}{(1+\sqrt{abc})^8}}.$$

Example 2. Assume $a,b,c > 0$ such that $a+b+c=1$. Prove inequality:

$$T = \sqrt[3]{\frac{2}{a}+9bc} + \sqrt[3]{\frac{2}{b}+9ca} + \sqrt[3]{\frac{2}{c}+9ab} \leq \sqrt[3]{\frac{6}{abc}+27}.$$

Proof: Because $f(x) = \sqrt[3]{x}, x > 0,$ is a concave function and from remark 1 we have

$$\sqrt[3]{\frac{2}{a}+9bc} + \sqrt[3]{\frac{2}{b}+9ca} + \sqrt[3]{\frac{2}{c}+9ab} \leq \frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{2}{a}+\frac{2}{b}+\frac{2}{c}+9(ab+bc+ca)}$$

or $\sqrt[3]{\frac{2}{a}+9bc} + \sqrt[3]{\frac{2}{b}+9ca} + \sqrt[3]{\frac{2}{c}+9ab} \leq \frac{3}{\sqrt[3]{3}}\sqrt[3]{(\frac{2}{abc}+9)(ab+bc+ca)}.$

Because $1 = (a+b+c)^2 \geq 3(ab+bc+ca)$. Hence we get $T \leq \sqrt[3]{\frac{6}{abc}+27}.$

Example 3. Supposed integer $n \geq 2$. Prove inequality:

$$\prod_{k=1}^{n}\frac{3^k-1}{3^{k-1}} \le (3-\frac{3}{2n}+\frac{3}{2n.3^n})^n.$$

Proof: Because $f(x)=\ln x, x>0,$ is a convex function and according theorem 2 we have $\frac{1}{n}(\sum_{k=1}^{n}\ln\frac{3^k-1}{3^{k-1}}) \le \ln[\frac{1}{n}(\sum_{k=1}^{n}\frac{3^k-1}{3^{k-1}})] = \ln(3-\frac{3}{2n}+\frac{3}{2.3^n}).$

Hence, we deduce inequality $\prod_{k=1}^{n}\frac{3^k-1}{3^{k-1}} \le (3-\frac{3}{2n}+\frac{3}{2n.3^n})^n.$

Student can synthesize the knowledge, ideas, generate the new knowledge from mathematical concepts and real life experience, and apply the ideas to construct some problems also revised when they find a hindrance.

Now, students would apply knowledge in a creative way. For example, consider a discrete random variable X with n possible values $x_1, x_{2,} ..., x_n$ In Equation of Theorem 2, we can choose $\alpha i = P(X = x_i) = PX(x_i)$. Then, the left-hand side of this equation becomes $g(EX)$ and the right-hand side becomes $E[g(X)]$. So we can prove the Jensen's inequality in this case. Using limiting arguments, this result can be extended to other types of random variables.

Jensen's Inequality: If $g(x)$ is a convex function on RX, and $E[g(X)]$ and $g(E[X])$ are finite, then

$$E[g(X)] \ge g(E[X]).$$

To use Jensen's inequality, we need to determine if a function *g* is convex. A useful method is the second derivative.

A twice-differentiable function *g*:*I*→R is convex if and only if $g''(x) \ge 0$ for all $x \in I$.

The thinking skills assessment helps teachers to identify whether students have the critical thinking and problem-solving skills which are essential for success in mathematical education. Assessment entails making comparisons and judgments. Teachers can encourage this level of thinking by asking students which of the mathematics knowledge they used worked the best.

3. Conclusion

To innovate teaching methods in Mathematics, it is important to innovate assessment. It is clear that teachers do not only need to know what to teach but also how to assess. Mathematics teachers in their efforts to help their pupils' success, strive to find the best ways to teach, so that the pupils are engaged in the learning process as well as developing student's thinking.

This research is one of the approaches to assess, identify students' high-order thinking in mathematics. It has described the student's high-order thinking. The difference of the levels is based on fluency, flexibility, and novelty in mathematical problem solving and problem posing. These levels are easier to apply in the mathematics classroom in Vietnam because teachers can examine the product of the task if their objective is to develop students' high-order thinking in mathematics.

We hope this research will teachers others to continue the research, verify, modify, or apply it. Thereby, improving the quality of not only teaching and learning Mathematics but education in Vietnam in general.

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