

THE DISTANCE BETWEEN THE CIRCUMCENTER AND ANY POINT IN THE PLANE OF THE TRIANGLE

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Abstract: In this article we give a metric relation which gives the distance between circumcenter to any point in the plane of the triangle.

Keywords: Circumcenter, Stewart's theorem, circumradius, Euler's Inequality.

1. Introduction

It is well-known that the perpendicular bisectors of the sides of a triangle meet at a single point, which is the center of the circumcircle [8],[9]. In this article we first give a metric relation which finds the distance between the circumcenter and any point in the plane of the triangle which allows us to prove some theorems related to circumcenter and in conclusion of the article we will prove the famous 'Euler's Inequality'.

2. Notations:

Let ABC be a triangle. We denote its side-lengths by $a=BC$, $b=AC$, $c=AB$, and angles by $\angle A$, $\angle B$ and $\angle C$ its semi perimeter by $s = \frac{a+b+c}{2}$, its area by Δ , Its

classical centers are the circumcenter S, the incenter I, the centroid G, and the orthocenter O. The nine-point center N is the midpoint of SO and the center of the nine-point circle, which passes through the side-midpoints and the feet of the three altitudes. The Euler Line Theorem states that G lies on SO with $OG : GS = 2 : 1$ and $ON : NG : GS = 3 : 1 : 2$. We write I_1, I_2, I_3 for the excenters opposite A, B, C, respectively, these are points where one internal angle bisector meets two external angle bisectors. Like I, the points I_1, I_2, I_3 are equidistant from the lines AB, BC, and CA, and thus are centers of three circles each tangent to the three lines. These are the excircles. The *classical radii* are the circumradius $R (= SA = SB = SC)$, the inradius r , and the exradii r_1, r_2, r_3 .

Let D, E and F are the foot of the cevians drawn through S from the vertices A, B and C to the opposite sides BC, CA and AB respectively.

The following formulae are well known

$$(a) \Delta = \frac{abc}{4R} = rs = r_1(s-a) = r_2(s-b) = r_3(s-c) = \sqrt{s(s-a)(s-b)(s-c)}$$

$$(b) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C = \frac{abc}{2R^3} = \frac{2\Delta}{R^2}$$

$$(c) \sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$$

$$(d) \sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$$

$$(e) -\sin 2A + \sin 2B + \sin 2C = 4 \sin A \cos B \cos C$$

$$(f) \cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

$$(g) \cos B + \cos C - \cos A = \frac{r_1}{R} - 1$$

$$(h) \cos A + \cos C - \cos B = \frac{r_2}{R} - 1$$

$$(i) \cos A + \cos B - \cos C = \frac{r_3}{R} - 1$$

$$(j) AI = \sqrt{r^2 + (s-a)^2} = \frac{r}{\sin \frac{A}{2}}, BI = \sqrt{r^2 + (s-b)^2} = \frac{r}{\sin \frac{B}{2}} \text{ and } CI = \sqrt{r^2 + (s-c)^2} = \frac{r}{\sin \frac{C}{2}}$$

$$(k) AI_1 = \sqrt{r_1^2 + s^2} = \frac{r_1}{\sin \frac{A}{2}}, BI_1 = \sqrt{r_1^2 + (s-c)^2} = \frac{r_1}{\cos \frac{B}{2}} \text{ and } CI_1 = \sqrt{r_1^2 + (s-b)^2} = \frac{r_1}{\cos \frac{C}{2}}$$

$$(l) \tan \frac{A}{2} = \frac{(s-b)(s-c)}{\Delta} = \frac{\Delta}{s(s-a)}$$

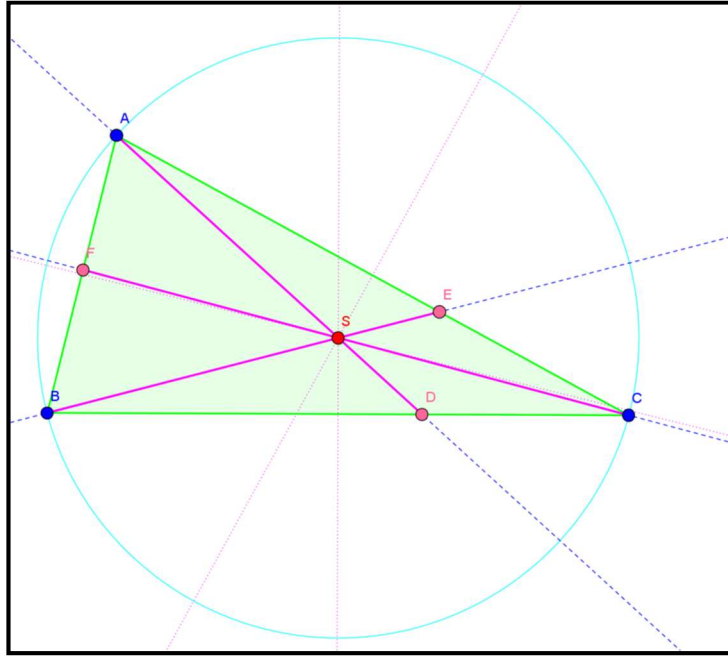
3. BASIC LEMMA'S

Lemma -1

If S is the circumcenter of triangle ABC (acute or right), the cevians through S from the vertices A, B and C intersect opposite sides BC, CA and AB at the points D, E and F respectively then

$$(1.1) \frac{BD}{DC} = \frac{\sin 2C}{\sin 2B}, \frac{CE}{EA} = \frac{\sin 2A}{\sin 2C} \text{ and } \frac{AF}{FB} = \frac{\sin 2B}{\sin 2A}$$

Proof:



Since S is circumcenter, we have $\angle ASB = 2C$ and $\angle ASC = 2B$

It implies $\angle BAD = 90 - C$ and $\angle CAD = 90 - B$

So by applying the sine rule for the triangles ABD and ACD,

We can show
$$\frac{BD}{DC} = \frac{\sin C \cos C}{\sin B \cos B} = \frac{\sin 2C}{\sin 2B}$$

Which proves the required results.

The above results are also true even if the triangle is obtuse, It is clear that the ratios are internal if the triangle is acute or right and they are external if the triangle is obtuse.

Lemma -2

If S is the circumcenter, then

$$(2.1) \quad SD = \frac{R \sin 2A}{\sin 2B + \sin 2C}, \quad SE = \frac{R \sin 2B}{\sin 2A + \sin 2C} \quad \text{and} \quad SF = \frac{R \sin 2C}{\sin 2A + \sin 2B}$$

$$(2.2) \quad AD = \frac{2A}{R(\sin 2B + \sin 2C)}, \quad BE = \frac{2A}{R(\sin 2A + \sin 2C)} \quad \text{and} \quad CF = \frac{2A}{R(\sin 2A + \sin 2B)}$$

Proof:

By lemma-1(1.1) we have $\frac{BD}{DC} = \frac{\sin 2C}{\sin 2B}$

$$\text{So } BD = \frac{a \sin 2C}{\sin 2C + \sin 2B} \quad \text{and} \quad CD = \frac{a \sin 2B}{\sin 2C + \sin 2B}$$

And Now for the triangle ADC, the line BSE acts as transversal

$$\text{So by Menelaus theorem we have } \frac{AE}{EC} \frac{CB}{BD} \frac{DS}{SA} = 1$$

By replacing the all known relations and by little algebra we can arrive at the required conclusions of (2.1)

And by using (b), and the fact $AD = AS + SD$, with little algebra we can prove the conclusions of (2.2) .

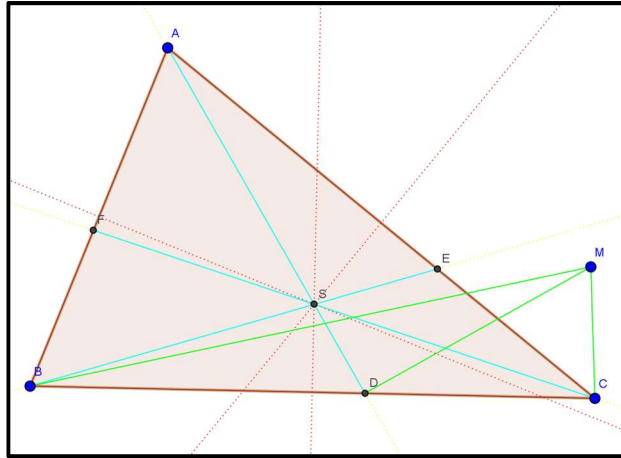
4.MAIN RESULT

THEOREM – 1

If S is the circumcenter of an acute or right triangle and M be any point in the plane of triangle, then

$$SM^2 = \frac{R^2}{2A} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2A)$$

Proof:



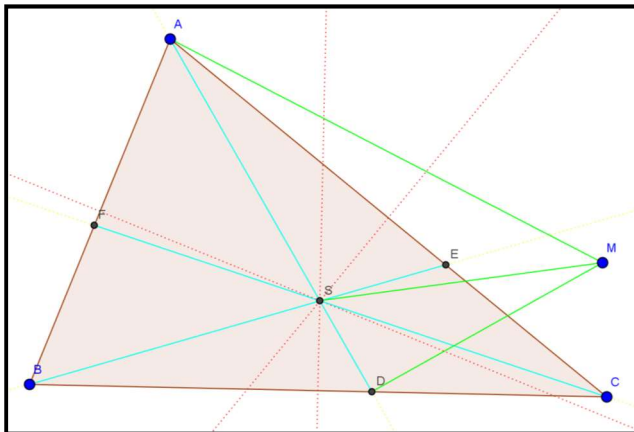
Let M be any point in the plane of triangle,

Now from triangle BMC, DM is cevian,

Hence by Stewart's theorem

$$\text{We have } DM^2 = \frac{BD \cdot CM^2}{BC} + \frac{CD \cdot BM^2}{BC} - BD \cdot DC$$

$$DM^2 = \frac{\sin 2C}{\sin 2C + \sin 2B} CM^2 + \frac{\sin 2B}{\sin 2C + \sin 2B} BM^2 - \frac{a^2 \sin 2B \sin 2C}{(\sin 2C + \sin 2B)^2}$$



Now from triangle ADM, SM is cevian,

Hence again by Stewart's theorem

$$\text{We have } SM^2 = \frac{AS \cdot DM^2}{AD} + \frac{SD \cdot AM^2}{AD} - AS \cdot SD$$

$$SM^2 = \frac{R^2(\sin 2C + \sin 2B)}{2\Delta} \left[\frac{\sin 2C}{\sin 2C + \sin 2B} CM^2 + \frac{\sin 2B}{\sin 2C + \sin 2B} BM^2 - \frac{a^2 \sin 2B \sin 2C}{(\sin 2C + \sin 2B)^2} \right] \\ + \frac{R^2 \sin 2A}{2\Delta} AM^2 - \frac{R^2 \sin 2A}{\sin 2C + \sin 2B}$$

Further simplification gives

$$SM^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2\Delta)$$

Hence proved.

The Theorem -1 is also true even if the triangle is obtuse.

5. APPLICATIONS

(I). Due to EULER the following are well known

$$(IA). SI^2 = R^2 - 2Rr$$

$$(IB). SI_1^2 = R^2 + 2Rr_1$$

Proof:

By theorem-1,

$$\text{We have } SM^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2\Delta)$$

Now let us fix M as I,

$$\text{So } SI^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AI^2 + \sin 2B \cdot BI^2 + \sin 2C \cdot CI^2 - 2\Delta)$$

$$\text{Using (j), } SI^2 = \frac{R^2}{2\Delta} \left(\sum \sin 2A \cdot \frac{r^2}{\sin^2 \frac{A}{2}} - 2\Delta \right)$$

Using (a), (b), (f) and (k) and by brute force computation

We can prove the required conclusion (IA)

In the similar manner if we fix M as I₁, and by computation we can arrive at the conclusion (IB).

(II). If X is any point on the circumcircle of triangle ABC, then

$$\sin 2A \cdot AX^2 + \sin 2B \cdot BX^2 + \sin 2C \cdot CX^2 = 4\Delta$$

Proof:

By Theorem-1,

$$\text{We have } SM^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2\Delta)$$

let us fix M as X,

$$SX^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AX^2 + \sin 2B \cdot BX^2 + \sin 2C \cdot CX^2 - 2\Delta)$$

and X is a point on the circumcircle of triangle so SX = R

$$\text{Hence } R^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AX^2 + \sin 2B \cdot BX^2 + \sin 2C \cdot CX^2 - 2\Delta)$$

Further simplification gives the desired result.

(III). Circumcenter lies on the line join of Weill's point and incenter.

Proof:

We have by Theorem -1,

$$SM^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AM^2 + \sin 2B \cdot BM^2 + \sin 2C \cdot CM^2 - 2\Delta)$$

Let us fix M as Weill's point (W_e), then

$$SW_e^2 = \frac{R^2}{2\Delta} (\sin 2A \cdot AW_e^2 + \sin 2B \cdot BW_e^2 + \sin 2C \cdot CW_e^2 - 2\Delta)$$

And by replacing the values of AW_e^2 , BW_e^2 and CW_e^2 and by some computation

$$\text{We can prove that } SW_e^2 = \left(1 + \frac{r}{3R}\right)^2 SI^2$$

$$\text{And we have } IW_e^2 = \left(\frac{r}{3R}\right)^2 SI^2$$

Now it is clear that $W_eI + IS = W_eS$

Hence circumcenter lies on the line join of Weill's point and incenter.

Another way of proving the above statement is available in [3].

In the similar argument whatever we adopted for proving the above statement by using Theorem-1, we can also prove that the circumcenter lies on the line join of different triangle centers Such as

The **Circumcenter** lies on the Line through the **Centroid** and the **Nine-Point Center**

The **Circumcenter** lies on the Line through the **Centroid** and the **Exeter Point**.

The **Circumcenter** lies on the Line through the **Centroid** and the **Schiffler Point**.

The **Circumcenter** lies on the Line through the **Centroid** and the **Gibert Point**.

The **Circumcenter** lies on the Line through the **Centroid** and the **Skordev Point**.

The **Circumcenter** lies on the Line through the **Orthocenter** and the **de Longchamps Point**.

The **Circumcenter** lies on the Line through the **Orthocenter** and the **Schiffler Point**.

The **Circumcenter** lies on the Line through the **Orthocenter** and the **Skordev Point**.

The **Circumcenter** lies on the Line through the **Nine-Point Center** and the **Orthocenter**.

The **Circumcenter** lies on the Line through the **Centroid** and the **Far-Out Point**

The **Circumcenter** lies on the Line through the **Mittelpunkt** and the **Orthocenter of the Extouch Triangle** Etc.

For Further details, see [7] and [9].

(IV) EULER'S INEQUALITY

If R is the Circumradius and r is the Inradius of a non-degenerate triangle then due to EULER we have an inequality referred as "Euler's Inequality" which states that $R \geq 2r$, and the equality holds when the triangle is Equilateral.

Proof:

We proved using theorem-1 $SI^2 = R^2 - 2Rr$

Now, Since the square of any real is non negative, we have $IS^2 \geq 0$ and it is clear that the equality holds when the triangle is equilateral.

It implies our desired Euler's Inequality $R \geq 2r$.

This ubiquitous inequality occurs in the literature in many different equivalent forms [1] and also Many other different simple approaches for proving this inequality are known. (some of them can be found in [3], [4], [5], [6], [12], [13]and [14])

For further study refer [9], [10], [11], [15].

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