PROBLEMS OLYMPICS IN GEOMETRY SPACE: EXPLORING CEVA AND MENELAUS WITH GEOGEBRA 3D

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ABSTRACT: In this article, we bring two classical theorems in plane geometry and its application in Olympic problems involving tetrahedra. Let's ascertain the possibilities of their discussion through the use/exploitation of the technology. Thus, we present the reader a learning scenario of a moment of proving and/or mathematical demonstration, with the support of GeoGebra 3D software.

KEYWORDS: Ceva's Theorem, Menelaus theorem, Olympic problems, tetrahedrons, GeoGebra 3D.

1 Introduction

The theorems of Ceva and Menelaus provides a more algebraic approach to concurrency and collinearity. It should be noted that these theorems have place for discussion guaranteed in the History of Mathematics. Hence, we can extract valuable implications for an approach that provides the structuring of scenarios learning which are very similar to those of mathematicians in the past, since the mobilization of tacit and intuitive reasoning was followed, when possible, an appropriate formalization. In this perspective, in several parts of the demonstrations, we add the visual value of geometric figures in order to extract and stimulate the perception ability of students.

2 Proposed problems and their solutions

The purpose of the proposed problems below is to show how their respective solutions are obtained by exploiting the GeoGebra.

Problem 1. (Austrian–Polish Mathematical Competition, 2002) Let $ABCD$ be a tetrahedron and let S be its center of gravity. A line through S intersects the surface of $ABCD$ in the points K and L. Prove that

$$\frac{1}{3} \leq \frac{\overline{KS}}{\overline{LS}} \leq 3.$$

Proof. Suppose K lies on the surface ABC whose intersection with the segment $\overline{DS}$ is the point G, i.e., $G =$ face $\Delta ABC \cap \overline{DS}$.

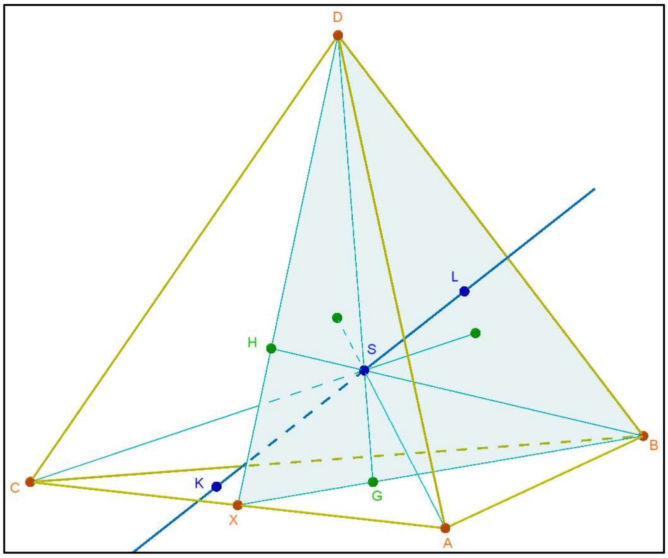

Figure 1. Tetrahedron $ABCD$ with center of gravity S.

Note in the plane BDX, the triangle GDX, such that $S \in \overline{GD}$ and $\overline{XB}$ It is an extension of the base $\overline{XG}$. In this triangle, the points H, S and B are collinear, then by Menelaus theorem, we have

$$\frac{\overline{XB}}{\overline{BG}} \cdot \frac{\overline{GS}}{\overline{SD}} \cdot \frac{\overline{DH}}{\overline{HX}} = 1 \quad (I)$$

We recall that the center of gravity (or centroid) of a tetrahedron is obtained by the intersection of the segments connecting each vertex of the tetrahedron to the center of gravity of their respective opposite side. So are valid the reasons

$$\frac{\overline{XB}}{\overline{BG}} = \frac{3}{2} \quad \text{and} \quad \frac{\overline{DH}}{\overline{HX}} = 2 \quad (II)$$

Therefore, it follows from (I) and (II) that

$$\frac{\overline{SD}}{\overline{GS}} = 3.$$

By proceeding analogously in the plane CDY we obtain, for Menelaus that

$$\frac{\overline{CS}}{\overline{RS}} = 3.$$

Moreover, by analyzing the ratio $\frac{\overline{KS}}{\overline{LS}}$, we find that it will be as higher as the distance between the points S and K, which occurs when K is the vertex C, and the smaller the distance between L and the point R. This gives us

$$\frac{\overline{KS}}{\overline{LS}} \leq \frac{\overline{CS}}{\overline{RS}} = 3.$$

The other inequality follows from the fact that the reason $\frac{\overline{KS}}{\overline{LS}}$ It will be as smaller the greater the distance between points S and L, which occurs when L is at the vertex D, and how much more near from S is K, in the case K coincides with the point G. Hence it follows that

$$\frac{1}{3} \leq \frac{\overline{KS}}{\overline{LS}} \leq 3,$$

as we wanted to prove. ■

Problem 2. (Vietnamese Mathematical Olympiad, 1998) Let be given a tetrahedron $ABCD$ whose circumcenter is O and $\overline{\mathrm{AA}_1}, \overline{\mathrm{BB}_1}, \overline{\mathrm{CC}_1}, \overline{\mathrm{DD}_1}$ diameters of the circumsphere of $ABCD$. Let A_0, B_0, C_0 e D_0 be the centroids of triangles BCD, CDA, DAB and ABC, respectively. Prove that:

(a) The segments $\overline{\mathrm{A}_0\mathrm{A}_1}, \overline{\mathrm{B}_0\mathrm{B}_1}, \overline{\mathrm{C}_0\mathrm{C}_1}$ and $\overline{\mathrm{D}_0\mathrm{D}_1}$ have a common point, denoted by F;

(b) the line that connects F with the midpoint of an edge is perpendicular to its opposite edge.

Proof. (a) Let $G = \overline{\mathrm{AA}_0} \cap \overline{\mathrm{BB}_0} \cap \overline{\mathrm{CC}_0} \cap \overline{\mathrm{DD}_0}$ the centroid of the tetrahedron $ABCD$. Let also $F = \overleftrightarrow{A_0A_1} \cap \overleftrightarrow{OG}$. By Menelaus theorem in the triangle ΔAOG we have

$$\frac{\overline{AO}}{\overline{A_1A}} \cdot \frac{\overline{A_0A}}{\overline{A_0G}} \cdot \frac{\overline{FG}}{\overline{FO}} = 1 \qquad (III)$$

But $\overline{\mathrm{AA}_1}$ is a diameter of the circumsphere, thus $\frac{\overline{A_1O}}{\overline{A_1A}} = \frac{1}{2}$, since O is the circumcenter of sphere. Furthermore, since G is the centroid of the tetrahedron, the previous question (the result of (I) and (II)) We have

$$\frac{\overline{AG}}{\overline{A_0G}} = 3,$$

And also,

$$\frac{\overline{A_0A}}{\overline{A_0G}} = \frac{\overline{AG} + \overline{A_0G}}{\overline{A_0G}} = \frac{\overline{AG}}{\overline{A_0G}} + \frac{\overline{A_0G}}{\overline{A_0G}} = 4.$$

Therefore, substituting the values obtained in (III), we conclude that $2\overline{FG} = \overline{FO}$. This means that the point F It is the point of reflection O with respect to the point G. Likewise, $\overline{B_0B_1}, \overline{C_0C_1}$ and $\overline{D_0D_1}$ pass through F. ■

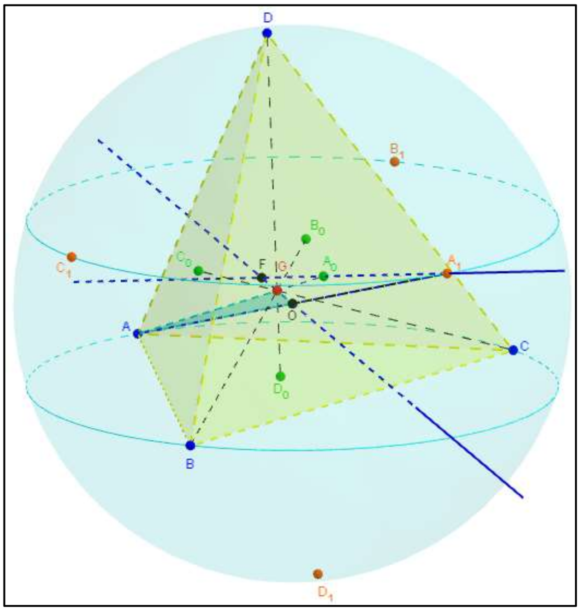

Figure 2. Sphere circumscribed in tetrahedron $ABCD$.

(b) Let M and N midpoints of edges $\overline{BC}$ and $\overline{AD}$, respectively. Since the point G is the midpoint of bimediana $\overline{MN}$, it follows that $NFMO$ is a parallelogram whose diagonals intersect at the point G. Thus, $\overline{MF} \parallel \overline{ON} \perp \overline{AD}$, i.e., the reverse lines $\overline{MF}$ and $\overline{AD}$ are orthogonal. Similarly, the lines that join the midpoints of $\overline{CD}, \overline{DB}, \overline{AB}, \overline{AC}$ and $\overline{AD}$ with F They are orthogonal to the edges $\overline{AB}, \overline{AC}, \overline{CD}, \overline{DB}$ and $\overline{BC}$, respectively. ■

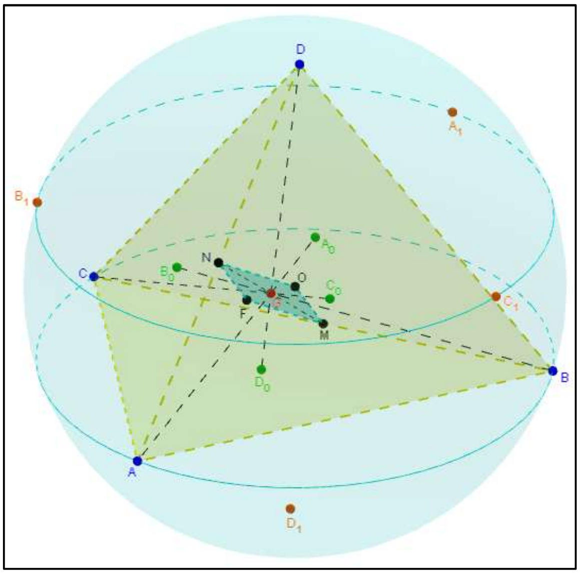

Figure 3. Parallelogram $NFMO$ inside on the tetrahedron $ABCD$.

The following lemma provides a generalization of Ceva theorem for the three-dimensional case and will be useful to the next problem.

Lemma (Ceva - space version). Let $ABCD$ be a tetrahedron such that $X \in \overline{AB}$, Y $\in \overline{BC}$, $Z \in \overline{CD}$ and $W \in \overline{DA}$. If it is equal (IV) below, then the plane AZB, BWC, CXD and DYA intersect exactly at one point.

$$\frac{\overline{XB}}{\overline{AX}} \cdot \frac{\overline{YC}}{\overline{BY}} \cdot \frac{\overline{ZD}}{\overline{CZ}} \cdot \frac{\overline{WA}}{\overline{DW}} = 1 \quad (IV)$$

Proof. Let M be the intersection point of the segments $\overline{XD}$, $\overline{WB}$ and N the intersection of the segments $\overline{BZ}$, $\overline{DY}$. Consider the segment from the point A to a point T in $\overline{BD}$ such that $\overline{AT}$ through the point M. By Ceva's theorem in the triangle ΔABD we have

$$\frac{\overline{AX}}{\overline{XB}} \cdot \frac{\overline{BT}}{\overline{TD}} \cdot \frac{\overline{DW}}{\overline{WA}} = 1. \quad (V)$$

Then, by multiplying Equations (IV) and (V), we obtain

$$\frac{\overline{BT}}{\overline{TD}} \cdot \frac{\overline{YC}}{\overline{BY}} \cdot \frac{\overline{ZD}}{\overline{CZ}} = 1.$$

This shows, by Ceva's theorem in the triangle ΔBCD, that $\overline{CT}$ through the point N. This result also leads to the conclusion that the segments $\overline{AN}$ and $\overline{CM}$ They are in the plane ATC, because the end points of each segment are in this plan. Furthermore, they intersect at a point P, since C it is on the opposite half-plane of M, having $\overline{AN}$ as the segment that separates the plane ATC into two half-planes.

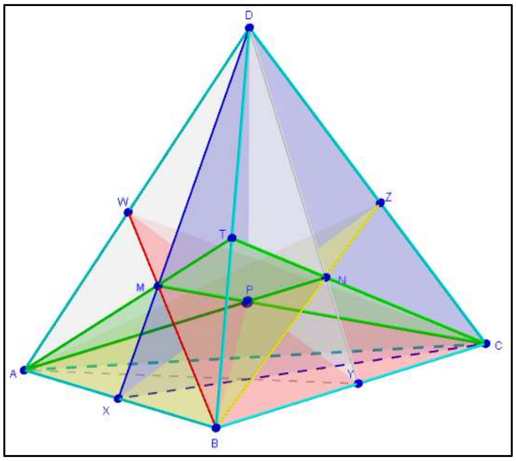

Figure 4. $\overline{AN}$ as intersection between the planes ADY and AZB and $\overline{CM}$ as intersection between the planes BWC and DCX.

Finally, as $\overline{AN}$ is the intersection between the planes ADY and AZB, $\overline{CM}$ is the intersection between the planes BWC and DCX, it follows that these four planes intersect in P. ■

Problem 3. (Spanish Mathematical Olympiad, 2011). Let A, B, C, D be four points in space not all lying on the same plane. The segments AB, BC, CD and DA are tangent to the same sphere. Prove that their four points of tangency are coplanar.

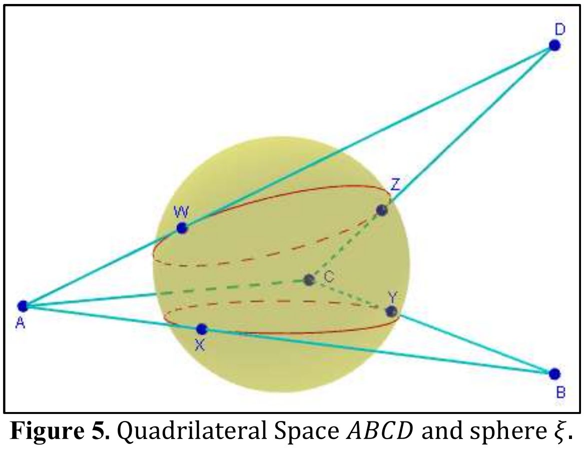

Figure 5. Quadrilateral Space $ABCD$ and sphere ξ.

Proof. Let X, Y, Z, W be the tangency points of the given sphere ξ with $AB,\ BC,\ CD$ and DA respectively. We will show that $\overline{XZ} \cap \overline{YW} = \{P\}$, since concurrent segments are coplanar. But $\overline{XZ} = \text{Plane}\, AZB \cap \text{Plane}\, DCX$ and $\overline{YW} = \text{Plane}\, BWC \cap \text{Plane}\, ADY$. Therefore, it is necessary to show that the planes AZB, BWC, CXD and DYA intersect at the point P. Thus, by figure 5, when the sphere ξ intersects the plane ACD determines a circle from which we can conclude that $\overline{ZD} \equiv \overline{DW}$ (to be tangent to the circle segments). Likewise, the sphere ξ intersects the plane ABC in another circle such that $\overline{XB} \equiv \overline{BY}$. Similarly the planes ABD and CBD also determine circles when intersect the sphere, so we get the following congruence: $\overline{WA} \equiv \overline{AX}$ and $\overline{YC} \equiv \overline{CZ}$. Hance we can write

$$\frac{\overline{XB}}{\overline{AX}} \cdot \frac{\overline{YC}}{\overline{BY}} \cdot \frac{\overline{ZD}}{\overline{CZ}} \cdot \frac{\overline{WA}}{\overline{DW}} = 1.$$

Therefore, the previous lemma, it is concluded that the planes AZB, BWC, CXD and DYA intersect at the point P and therefore, points X, Y, Z, W are coplanar. ■

Problem 4. (Tournament of Towns, 2009). Every edge of a tetrahedron is tangent to a given sphere. Prove that the three line segments joining the points of tangency of the three pairs of opposite edges of the tetrahedron are concurrent.

Proof. Let $ABCD$ be a tetrahedron such that the edges BC, CA, AB, DB, DC and DA are tangents to a sphere ξ in points P, Q, R, S, T and U, respectively. According to problem 3 above, we have points P, T, U and R they are coplanar and similarly the points R, S, T and Q they are also coplanar. So, $PU \cap RT \neq \emptyset$ and $QS \cap RT \neq \emptyset$. Furthermore, the segments PU, QS and RT are projected in the plane ABC exactly the segments AP, BQ and CR, cevianas of triangle ΔABC that concurrent at the Gergonne point G. Therefore the intersections $PU \cap RT$ and $QS \cap RT$ coincide along of DG. This shows that PU, QS and RT are concurrents. ■

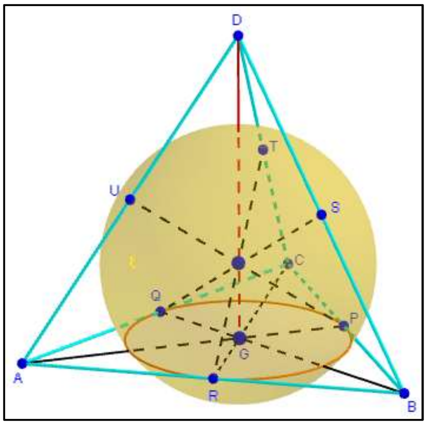

Figura 6. Tetrahedron $ABCD$ and the sphere ξ showing that PU, QS and RT are concurrents.

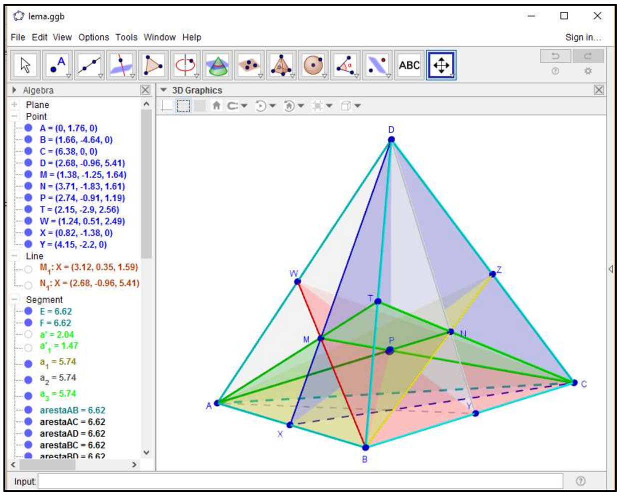

Figure 7. Indication of commands employees in geometric construction that generalizes the Ceva theorem.

To conclude this section, we show the software home screen that show some of constructions required by lemma that generalizes the Ceva theorem. We note, however, that the required syntax of software requires some didactic effort of teacher.

3 Conclusion

In the steps that involve elaboration of solutions in Olympic problems is important to find an optimal balance during time spent on the visualization and theoretical proofing. Besides understanding the theorems, our students need to find good proofs, thus it is need to manipulate/explore geometrical constructions, hence the importance the dynamic geometry systems that bring rapid progress in the involved reasoning.

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